On the Property and Configuration of the Short Time DFT Hilbert Transformers

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Short Time DFTを用いたHilbert変換器の特性と構成

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Hilbert transformers have been previouly studied with putting emphasis upon signal processing [1], as they are inevitable in preventing such modulation as single side band (ab. in SSB) or RZ SSB [2] from increasing frequency bandwidth over radio channels.

By adopting instanteneous spectrum analysis, namely "ST-DFT" (short time Discrete Fourier Transform, ab. in ST-DFT), a nobel realization of Hilbert transformers is proposed in this paper and sucessfully examined to be free from errors via computer simulations.

1. INTRODUCTION

It is unfortunate that existing Hilbert transformers have been realized to possess phase shifting function by approximate $\pi/2$ radian from the point of view for all-pass filter. On the other hand, ST-DFT Hilbert transformers proposed here are based on the instantanuous spectrum analysis [3]. Where the instantenuous spectrums are given, Hilbert transform is precisely carried out via merely exhanging real or imaginary part of these spectrums with each other. Hilbert transformed output signals are, therefore, synthesized via inverse ST-DFT (ab. in ST-IFT) from the interchanaged instantenuous spectrums.

A circuitry configuration of the ST-DFT Hilbert transformer is categorized into three major blocks, namely, I instantanuous spectrum analyzer, II frequency domain Hibert transformer, III output signal synthesizer. It is easy to understand that the ST-DFT Hilbert transform is ideal and free from any distortion, while the instantanuous spectrum analysis is performed without any errors.

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2. INSTANTANEOUS SPECTRUM ANALYSIS AND SYNTHESIS

Consider first how the ST-DFT gives the instantaneous spectrum and what characteristic it possesses. Let instantaneous spectrum $\Phi(n)$ at sampling time n be described by,

$$\Phi(n) = \{ \phi_0(n) \quad \phi_1(n) \quad \phi_2(n) \cdot \cdot \cdot \quad \phi_{N-1}(n) \}^T. \tag{1}$$

Where, $\phi_K(n)$ is a spectrum component at frequency index k of $\Phi(n)$ given as,

$$\phi_K(n) = \Sigma_{r=-\infty}^{\infty}\ x(r)\ h(n-r)\ W_N^{-rk}, \qquad (2)$$

$W_N^{-rk} = \exp\{-j(2\pi rk/N)\}$,
integer k is $0 \leq k < N$.

Here, $x(r)$ is an input data at sampling time r. W_N^{-rk} is the same operator defined as that of existing DFT.

Existence of ST-DFT requires that output signal $y(n)$ at time n is precisely produced from the instantenuous spectrum $\Phi(n)$ via ST-IFT. That is,

$$N \cdot y(n) = \Sigma_{k=0}^{N-1} \phi_K(n) W_N^{nk}, \qquad (3)$$

$W_N^{nk} = \exp\{j(2\pi nk/N)\}$.

Here, W_N^{nk} is the same operator as existing inverse DFT.

The ST-IFT denoted by eq.3 requires that the window function h(*) of eq.2 holds condition $y(n) = x(n)$. Substituting eq.2 into eq.3 and exchanging the summation order for variables k and r, it gives eq.4.

$$N \cdot y(n) = \Sigma_{k=0}^{N-1}\ \{\Sigma_{r=-\infty}^{\infty}\ x(r)\ h(n-r)\ W_N^{-rk}\}\ W_N^{nk}$$

$$= \Sigma_{r=-\infty}^{\infty}\ x(r)\ h(n-r)\ \ \{\Sigma_{k=0}^{N-1} W_N^{(n-r)k}\}\ . \qquad (4)$$

The summation for variable k takes non zero value by N, only if $n-r = 2Nq$. Here, q is an integer. This gives a restrict condition to the window function as follows,

$$h(p) = \begin{cases} 1, & \text{if } p = 0 \\ 0, & \text{if } p = 2Nu, \end{cases} \qquad (5)$$

u is non zero integer.

For example, an N frame length Nyquist window function truncated with 2m frame number h(p),

$$h(p) = \sin(p\pi/N) / (p\pi/N), \qquad (6)$$

$-mN \leq p \leq mN$,

are able to satisfy eq.5. Hereafter, the truncated Nyquist will be employed for the present as a window function in the ST-DFT.

3. PRINCEPLE OF ST-DFT HILBERT TRANSFORMERS

Figure 1 shows the processing outline in the ST-DFT Hilbert transformer. As being discussed in previous section, input signal $x(r)$ is at first in the ST-DFT Hilbert transformer analyzed by ST-DFT to yield instantaneous spectrum $\Phi(n)$. Secondly, both real and imaginary parts of each component of the instantaneous spectrum are exchanged with each other to get Hilbert transformed spectrum $\hat{\Phi}(n)$. Output signal $\hat{y}(n)$ is finally synthesized through ST-IFT from the transformed spectrum.

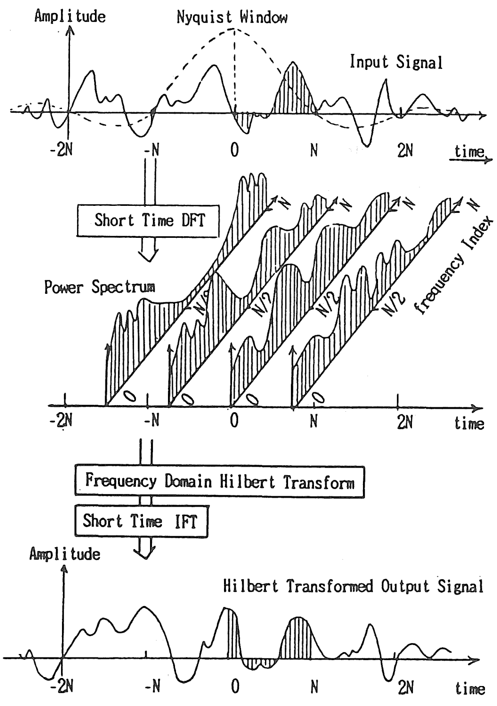


Fig.1 Processing Outline in the ST-DFT Hilbert Transform

Both instantanuous spectrum analysis and phase shifting being combined into one operator, the frequency domain Hilbert transform operator $\widehat{W}_N^{-rk}$ is given as follows,

$$\widehat{W}_N^{-rk} = \begin{cases} \exp\{-j(2\pi rk/N + \pi/2)\}, & \text{if } 0< k< N/2 \\ 0, & \text{if } k=0,\ N/2 \\ \exp\{-j(2\pi rk/N - \pi/2)\}, & \text{if } N/2< k< N. \end{cases} \quad (7)$$

Here, j is complex unit, $j=\sqrt{-1}$.

Proof:Existence of Frequency Domain Hilbert Transform Operator

Constant $\pi/2$ on the first row of eq.7 gives features of phase shifting function by $\pi/2$ radian to the k-*th* component $\phi_k(n)$ of $\Phi(n)$. This also clearly shows that the frequency domain Hilbert transform is free from amplitude distortion, because operator $\widehat{W}_N^{-rk}$ consists of single complex exponential function only with pure imaginary variables.

Both the second and third rows of eq.7 satisfy that $\hat{y}(n)$ exists phisically. Substituting $\hat{\phi}_k(n)$ with $\phi_k(n)$ of eq.3, $\hat{y}(n)$ is given as follows,

$$N\cdot\hat{y}(n)=\Sigma_{k=0}^{N-1}\hat{\phi}_k(n)W_N^{nk}$$

$$=\hat{\phi}_0(n)W_N^{0}+\hat{\phi}_{N/2}(n)W_N^{nN/2} + \Sigma_{k=1}^{N/2-1}\{\hat{\phi}_k(n)W_N^{nk}+\hat{\phi}_{N-k}(n)W_N^{n(N-k)}\} \quad (8)$$

In eq.8, W_N^0 or $W_N^{nN/2}$ takes pure real 1 or 1(-1), respectively. If $\phi_0(n)$ or $\phi_{N/2}(n)$ is non zero, the corresponding transformed component $\hat{\phi}_0(n)$ or $\hat{\phi}_{N/2}(n)$ takes pure imaginary value and the output $\hat{y}(n)$ diversifies to complex. So $\widehat{W}_N^{-rk}$ has to vanish at k=0 and N/2 as defined in eq.7.

Since $W_N^{n(N-k)}$ is given by complex conjugate with W_N^{nk}, i.e. $W_N^{n(N-k)}=\overline{W}_N^{nk}$, every bracketted term of eq.8 takes a pure real number, if and only if $\hat{\phi}_{N-k}(n)$ is complex conjugate with $\hat{\phi}_k(n)$. In practice, $\hat{\phi}_{N-k}(n)$ is given as

$$\hat{\phi}_{N-k}(n)$$

$$=\Sigma_{r=-\infty}^{\infty} x(r)h(n-r)\exp[-j\{2\pi r(N-k)/N - \pi/2\}]$$

$$=\Sigma_{r=-\infty}^{\infty} x(r)h(n-r)\exp[\,j\{2\pi rk/N + \pi/2\}]$$

$$=\overline{\hat{\phi}}_k. \quad (9)$$

Therefore, the output $\hat{y}(n)$ becomes pure real number as follows,

$$N\cdot\hat{y}(n)=\Sigma_{k=1}^{N/2-1}\{\hat{\phi}_k(n)W_N^{nk}+\hat{\phi}_{N-k}(n)W_N^{n(N-k)}\}$$

$$=\Sigma_{k=1}^{N/2-1}\{\hat{\phi}_k(n)W_N^{nk}+\overline{\hat{\phi}}_k(n)\overline{W}_N^{nk}\}$$

$$=\Sigma_{k=1}^{N/2-1} 2Real\{\hat{\phi}_k(n)W_N^{nk}\}, \quad QED. \quad (10)$$

4. CIRCUITRY CONFIGURATION AND ITS UNIT SAMPLE RESPONSE

Figure 2 shows a primitive block diagram of ST-DFT Hilbert transformers. ST-DFT Hilbert transformers are categorized into three major functional blocks.

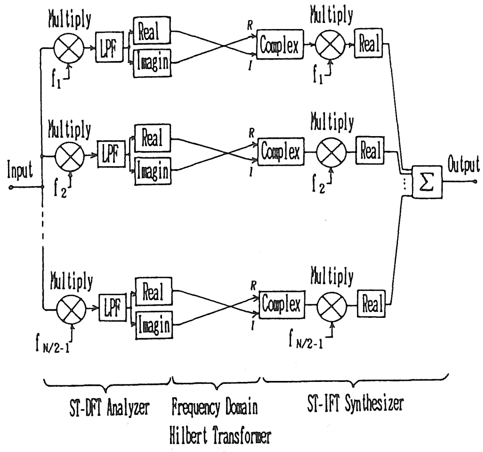
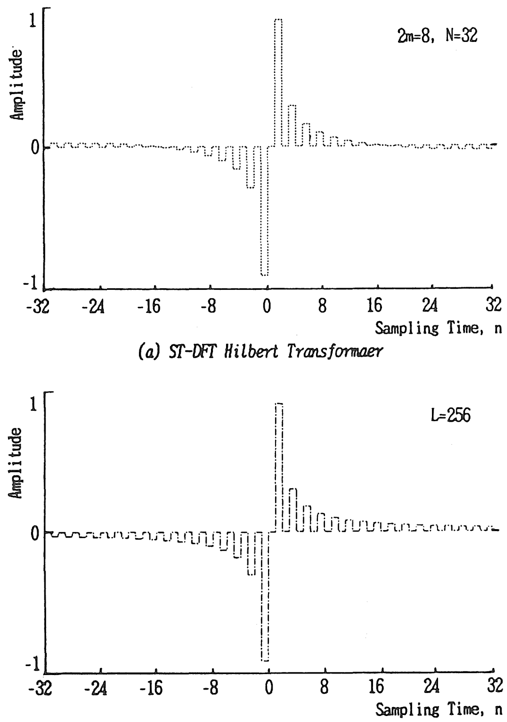


Fig.2 Configuration of the ST-DFT Hilbert Transformer

The first functional block plays a role of ST-DFT analayzers and consists of N/2-1 channel modules in which every component $\phi_k(n)$ of the instantaneous spectrum $\Phi(n)$ is yieled, here k=1, 2, ..., N/2-1. Inner product of x(n) and W_N^{-rk} in eq.2 is performed of modulating the input x(n) with complex carrier W_N^{-rk} of $2\pi k/N$ normarized angular frequency. Convolution $\{x(r)W_N^{-rk}\}$ and h(n-r) in eq.2 is also interpreted as low-pass filtering for modulated signal $\{x(r)W_N^{-rk}\}$.

The second block acts as a Hilbert transformer on the frequency domain, which exchanges the real with the imaginary part of $\phi_k(n)$. This block is dominant in function, however, its circuitry configuration is so simple as it only posesses two crossing wires as shown in fig.2. The first and second blocks are practically combined together in frequency index wise to get $\hat{\phi}_k(n)$ directly by adopting $\hat{W}_N^{-rk}$ instead of W_N^{-rk} during the modulation.

The last is synthesizer which emploies ST-IFT to produce time domain signal from Hilbert transformed spectrum $\hat{\Phi}(n)$.
In similar to the first block, ST-IFT is performed of modulating Hilbert transformed spectrum component $\hat{\phi}_k(n)$ with complex carrier W_N^{nk} of $2\pi k/N$ normarized angular frequency.



(a) ST-DFT Hilbert Transformaer

(b) Minmax FIR Hilbert Transformer

Fig.3 Comparison of unit sample response between the ST-DFT and minmax Hilbert transformers

An unit sample response Is(n) of the ST-DFT Hilbert transformer is given in eq.11 by using eqs.6, 7 and 8 when the unit sample x(n) is taken to be x(0)=1.

$$Is(n)=\begin{cases}\dfrac{2}{N}\dfrac{\sin(2\pi n/N)}{\{1-\cos(2\pi n/N)\}}\dfrac{\sin(\pi n/N)}{\pi n/N} \\ \qquad =\dfrac{2\cos(\pi n/N)}{\pi n}, & \text{if n is odd} \\ 0, & \text{if n is even}\end{cases} \qquad (11)$$

Eq.11 shows the ideal unit sample response of the Hilbert transformer, because the frequncy domain Hilbert transform operator of eq.7 does exclude all of the processing error and because the Nyquist window with infinite length acts as an ideal low-pass filter as shown in the factor of Nyquist window $\sin(\pi n/N)/(\pi n/N)$.

Equation 12 gives the unit sample response Im(n) of Rabinor's minmax FIR Hilbert (ab. in minmax) transformer[1].

$$Im(n)=\begin{cases}\dfrac{2\sin^2(\pi n/2)}{\pi n}=\dfrac{1-\cos(\pi n)}{\pi n} \\ \qquad =\dfrac{2}{\pi n}, & \text{if n is odd} \\ 0, & \text{if n is even}\end{cases} \qquad (12)$$

It is clearly shown in eqs.11 and 12 that the ST-DFT Hilbert transformer enhances minmax transformer. That is,

$$\lim_{N\to\infty} Is(n) = \lim_{N\to\infty} \frac{2\cos(\pi n/N)}{\pi n} = \frac{2}{\pi n} = Im(n), \tag{13}$$

here, n is odd.

5. RESULTS OF COMPUTER SIMULATIONS

The ST-DFT Hilbert transformer is substantiated through computer simulations both of unit sample response and phase shifting. All the simulations were carried on the CRAY X-MP of AIT to avoid roundoff error from calculation.

Figure 3 shows evident differnce between unit sample response of ST-DFT Hilbert transformer (a) and that of minmax one (b). Here, the length L of FIR filter of minmax Hilbert transformer is set to be 256. The frame number 2m and frame length N of the ST-DFT transformer are set to be 8 and 32. The total length of the ST-DFT window h(*), therefore, becomes 256 (2mN=8X32) the same to minmax FIR filter.

As shown in fig.3(a), the unit sample response of ST-DFT transformer features of oscilation of its envelope being coincident with that of minmax transformer shown in fig.3(b). This oscilation of the envelope is induced with the cos(πn/N) factor of eq.11.

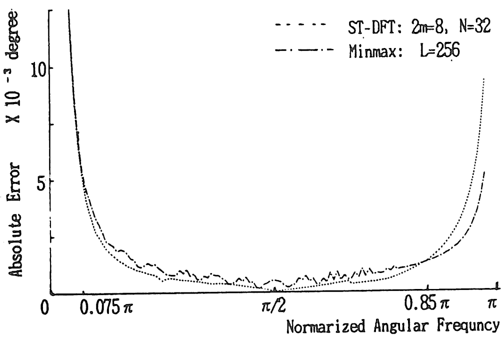


Fig.4 Phase shifting absolute error of the ST-DFT and minmax Hilbert transformers

That is, the response of ST-DFT transformer oscilates on all the 2mN sampling time with N period. On the other hand, minmax transformer response is non periodic on all the L sampling time.

This periodicity of the ST-DFT transformer improves phase shifting function in accuracy. Figure 4 shows phase shifting errors both of the ST-DFT and minmax transformer for tonal input signals. Where, the length 2mN and L are equally taken to be 256 and output signals are both analyzed by 256 frame length ST-DFT.

The ST-DFT transformer error indicated in fig.4 by dotted line is observed as 2m=8 and N=32. The ST-DFT transformer does not exceed beyond the phase shifting error of minmax transformer also shown in fig.4 by chained line on the frequency domain of less than 0.85 π (= 3.4kHz of 8kHz sampling) and guarantees that the phase shifting error is within 5x10 $^{-3}$ degree for the practical frequency domain of more than 0.075π (=0.3kHz of 8kHz sampling) less than 0.85π.

CONCLUSION

Putting emphasis on the instantaneous spectrum signal processing, a noble Hilbert transformer was discussed through its circuitry configuration, unit sample response and phase shifting function. In spite of adopting a primitive truncated Nyquit for the significant window h (*), ST-DFT Hilbert transformer can exceed Rabiner's minmax one in both phase shifting accuracy and rapidness of transient response. Farther studies will improve such primitive instantanuous spectrum signal processing as ST-DFT, ST-DFT Hilbert transformer, etc.

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