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＜査読付論文＞

Planar open Riemann surfaces and holomorphic approximation

単葉型開リーマン面と正則近似

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Abstract　An open Riemann surface R is planar if and only if for every domain G in R the condition that G satisfies the strong disk property in R implies the condition that G is holomorphically Runge in R.

1. Introduction

First, we prove that for every open Riemann surface R such that $1 \leq g(R) \leq +\infty$ there exists a relatively compact annular domain G in R such that G is not holomorphically Runge in R whereas G satisfies the strong disk property in R (see Theorem 3.1). As a corollary, an open Riemann surface R is planar if and only if for every domain G in R the condition that G satisfies the strong disk property in R implies the condition that G is holomorphically Runge in R, which answers Abe-Nakamura [5, Problem 3.5] (see Corollary 3.2).

Next, we prove that a domain G in an arbitrary open Riemann surface R satisfies the strong disk property in R if and only if the canonical homomorphism $\pi_1(G) \to \pi_1(R)$ is injective (see Theorem 4.2), the proof of which is based on the argument in the proof of Abe [2, Theorem 5].

Alternative proofs for both Corollary 3.2 and Theorem 4.2 based mainly on the theory of functions in one complex variable are also presented in the paper [6].

2. Preliminaries

Complex manifolds are always supposed to be second countable. We denote by $\mathscr{O}(R)$ the set of holomorphic functions on R. A complex manifold R is said to be *Stein* if the following two conditions are satisfied:

- R is *holomorphically separable*, that is, for any two points $p, q \in R$, $p \neq q$, there exists $f \in \mathscr{O}(R)$ such that $f(p) \neq f(q)$.
- R is *holomorphically convex*, that is, for every compact set K of R, the *holomorphically convex hull* $\hat{K}_R$ of K in X is also compact, where

$$\hat{K}_R := \left\{ x \in R \mid \left| f(x) \right| \leq \left\| f \right\|_K \text{ for every } f \in \mathscr{O}(R) \right\}.$$

An open set D of a complex manifold R is said to be (*holomorphically*) *Runge* in R if for every $f \in \mathscr{O}(D)$, for every compact set K of D, and for every $\varepsilon > 0$, there exists $h \in \mathscr{O}(R)$ such that $\left\| f - h \right\|_K < \varepsilon$.

A connected complex manifold of dimension 1 is said to be a *Riemann surface* and a noncompact Riemann surface is said to be an *open Riemann surface*. By Behnke-Stein [7], every open Riemann surface is Stein. We have the following characterizations of a Runge open set of an open Riemann surface, which is also due to Behnke-Stein [7] (see Mihalache [11]).

Theorem 2.1 (Behnke-Stein)**.** *Let R be an open Riemann surface and D an open set of R. Then, the following three conditions are equivalent.*

(1) *D is Runge in R.*

(2) *The canonical homomorphism $H_1(D, \mathbb{Z}) \to H_1(R, \mathbb{Z})$ is injective.*

(3) *No connected component of $R \setminus D$ is compact.*

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Let $\mathbb{U} := \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ be the *unit disk* in $\mathbb{C}$. An open set D of a complex manifold R is said to satisfy the *strong disk property* in R if D satisfies the condition that if $\lambda : \overline{\mathbb{U}} \to R$ is a continuous map holomorphic on $\mathbb{U}$ such that $\lambda(\partial\mathbb{U}) \subset D$, then $\lambda(\overline{\mathbb{U}}) \subset D$. As is easily shown, we have the following proposition (see Abe [2, Proposition 1] and Abe-Nakamura [5, Proposition 2.6]).

Proposition 2.2. *Let R be a Stein manifold and D an open set of R. If every connected component of D is Runge in R, then D satisfies the strong disk property in R.*

A connected open set of a complex manifold R is said to be a *domain* in R. An open Riemann surface R is said to be *planar* if R is biholomorphic to a domain in $\mathbb{C}$. If R is a planar open Riemann surface, then the converse of Proposition 2.2 is true, that is, we have the following proposition (see Abe-Nakamura [5, Theorem 3.3]).

Proposition 2.3. *Let R be a planar open Riemann surface and D an open set of R. Then, the following two conditions are equivalent.*

(1) *D satisfies the strong disk property in R.*

(2) *Every connected component of D is Runge in R.*

3. Planar open Riemann surfaces

A domain G in a Riemann surface R is said to be a *normal domain* in R if G is relatively compact in R, the boundary ∂G of G consists of finitely many simple closed analytic paths in R, and no connected component of $R \setminus G$ is compact (see Nakai [12, p. 60]). We denote by $g(R)$ the *genus* of a Riemann surface R. We refer to Nakai [12, pp.118–119] for the definition of the genus of an open Riemann surface. Then, an open Riemann surface R is planar if and only if $g(R) = 0$.

Theorem 3.1. *Let R be an open Riemann surface such that $1 \le g(R) \le +\infty$. Then, there exists a relatively compact annular domain G in R such that G is not Runge in R while G satisfies the strong disk property in R.*

Proof. Take a normal domain S in R such that $1 \le g(S) < +\infty$. Let $\{\boldsymbol{a}_i, \boldsymbol{b}_i\}_{i=1}^{g}$, where $g := g(S)$, be a canonical homology basis of S modulo ∂S (see Nakai [12, p. 118]). There exist a compact Riemann surface S^* of genus g and an open disk $W = \{|z| < 1\}$, where z is a local coordinate of S^* defined near $\overline{W}$, such that S is a domain in S^* and $K := S^* \setminus S \subset W$ (see Nakai [12, pp. 187–189]). Then, $H := S \cap W$ and $E := S^* \setminus \overline{W}$ are nonempty domains in S. We may further assume that $\{\boldsymbol{a}_i, \boldsymbol{b}_i\}_{i=1}^{g} \subset E$. Take a number $\rho \in (0,1)$ such that $K \subset \{|z| < \rho\}$ and let $G := \{\rho < |z| < 1\}$. Since $S \setminus H = S^* \setminus W$ is a compact connected component of $S \setminus G$, the domain G is not Runge in S by Theorem 2.1 and, therefore, G is not Runge either in R.

Let $\lambda : \overline{\mathbb{U}} \to R$ be a continuous map holomorphic on $\mathbb{U}$ such that $\lambda(\partial\mathbb{U}) \subset G$. Since S is Runge in R, we have $\lambda(\overline{\mathbb{U}}) \subset S$ by Proposition 2.2. Suppose that $E \subset \lambda(\mathbb{U})$. Then, the map $\lambda : \mathbb{U} \to S$ is open and all fibers $\lambda^{-1}(x)$, $x \in \lambda(\mathbb{U})$, are discrete in $\mathbb{U}$. Since we can verify that $\lambda : \lambda^{-1}(E) \to E$ is proper, the map $\lambda : \lambda^{-1}(E) \to E$ is finite (see Grauert-Remmert [9, p. 175]). It follows that there exists $b \in \mathbb{N}$ such that $\lambda : \lambda^{-1}(E) \to E$ is a b-sheeted analytic covering of E (see Grauert-Remmert [9, pp. 135–136]). Let T be a critical locus of this analytic covering. Let $\gamma : I \to E$, where $I = [0,1]$, be an arbitrary closed path in E. Since T is a discrete closed set of E, the set $\gamma(I) \cap T$ is finite. Therefore, by deforming γ slightly, we have a closed path $\beta : I \to E \setminus T$ which is homotopic to γ in E. Let $a := \beta(0) = \beta(1)$. Take an arbitrary point $c_0 \in \lambda^{-1}(a)$. Since $\lambda : \lambda^{-1}(E \setminus T) \to E \setminus T$ is an unramified covering of $E \setminus T$, there exists a path $\tilde{\beta}_1 : I \to \lambda^{-1}(E \setminus T)$ such that $\lambda \circ \tilde{\beta}_1 = \beta$ and $\tilde{\beta}_1(0) = c_0$. Let $c_1 := \tilde{\beta}_1(1)$. Then, we have $\lambda(c_1) = \lambda(\tilde{\beta}_1(1)) = \beta(1) = a$. By induction, there exist points $c_1, c_2, \ldots, c_b \in \lambda^{-1}(a)$ and paths $\tilde{\beta}_\nu : I \to \lambda^{-1}(E \setminus T)$ such that $\lambda \circ \tilde{\beta}_\nu = \beta$, $\tilde{\beta}_\nu(0) = c_{\nu-1}$, and $\tilde{\beta}_\nu(1) = c_\nu$ for every $\nu = 1, 2, \ldots, b$. Since $\#\lambda^{-1}(a) = b < +\infty$, there exist nonnegative integers k and l such that $0 \le k < l \le b$ and $c := c_k = c_l$. Let $\tilde{\beta} := \tilde{\beta}_{k+1} \cdot \tilde{\beta}_{k+2} \cdot \cdots \cdot \tilde{\beta}_l : I \to \lambda^{-1}(E \setminus T)$ be the closed path which joins paths $\tilde{\beta}_{k+1}, \tilde{\beta}_{k+2}, \ldots, \tilde{\beta}_l$ successively. Then, we have $\lambda \circ \tilde{\beta} = \beta^m$, where $m := l - k \ge 1$. Since $\mathbb{U}$ is simply connected, there exists a homotopy $\tilde{\eta} : I \times I \to \mathbb{U}$ such that $\tilde{\eta}(0,t) = \tilde{\beta}(t)$ and $\tilde{\eta}(1,t) = \tilde{\eta}(s,0) = \tilde{\eta}(s,1) = c$ for every $s, t \in I$. Let $\eta := \lambda \circ \tilde{\eta} : I \times I \to \lambda(\mathbb{U})$. Then, we have $\eta(0,t) = \beta^m(t)$ and $\eta(1,t) = \eta(s,0) = \eta(s,1) = a$ for every $s, t \in I$. Therefore, β is homotopic to a constant path in $\lambda(\mathbb{U})$ because $\pi_1(\lambda(\mathbb{U}))$ is torsion free (see Napier-Ramachandran [13, p. 226]). It follows that $[\gamma] = [\beta] = 0$ in $H_1(\overline{S}, \mathbb{Z})$, which is a contradiction, for example, for $\gamma := \boldsymbol{a}_1$. Thus, we proved that $E \not\subset \lambda(\mathbb{U})$.

Take an arbitrary $r \in E \setminus \lambda(\mathbb{U})$. Then, $P := S^* \setminus \{r\}$ is a noncompact domain in S^*, $\overline{W} \subset P$, and $P \setminus W = (S^* \setminus W) \setminus \{r\}$ is not compact. We can also verify that $P \setminus W$ is connected. Therefore, W is Runge in P by Theorem 2.1. Since $\lambda(\partial\mathbb{U}) \subset G \subset W$ and $\lambda(\overline{\mathbb{U}}) \subset P$, we have $\lambda(\overline{\mathbb{U}}) \subset W$ by Proposition 2.2. It follows that $\lambda(\overline{\mathbb{U}}) \subset W \cap S = H$. Since the set $H \setminus G = \{|z| \le \rho\} \setminus K$ is connected and noncompact, the domain G is Runge in H by Theorem 2.1. Therefore, we have $\lambda(\overline{\mathbb{U}}) \subset G$ by Proposition 2.2. Thus, we proved that G satisfies the strong disk property in R. □

By Proposition 2.3 and by Theorem 3.1, we have the following characterization of a planar open Riemann surface in the class of the open Riemann surfaces.

Corollary 3.2. *Let R be an open Riemann surface. Then, the following two conditions are equivalent.*

(1) *R is planar.*

(2) *For every domain G in R, the condition that G satisfies the strong disk property in R implies the condition that G is Runge in R.*

4. A topological criterion

An open set D of a complex manifold R is said to be *meromorphically $\mathcal{O}(R)$-convex* if for every compact set K of D the set ${}_H K_R \cap D$ is also compact, where

$$ {}_H K_R := \{x \in R \mid f(x) \in f(K) \text{ for every } f \in \mathcal{O}(R)\} $$

is the *meromorphically convex hull* of K in R (see Hirschowitz [10], Colţoiu [8], Abe–Furushima [4], and Abe [1, 2, 3]). By the proof of Abe [2, Theorem 5], we have the following theorem.

Theorem 4.1. *Let R be a Stein manifold and G a meromorphically $\mathcal{O}(R)$-convex domain in R. Assume that the canonical homomorphism $\pi_1(G) \to \pi_1(R)$ is injective. Then, G satisfies the strong disk property in R.*

We have the following characterization of the strong disk property for a domain G in an arbitrary open Riemann surface R.

Theorem 4.2. *Let R be an open Riemann surface and G a domain in R. Then, the following two conditions are equivalent.*

(1) *G satisfies the strong disk property in R.*

(2) *The canonical homomorphism $\pi_1(G) \to \pi_1(R)$ is injective.*

Proof. (1) → (2). Let $\pi : Z \to R$ be the universal covering of R, where $Z = \mathbb{C}$ or $Z = \mathbb{U}$. Take an arbitrary closed path $\gamma : I \to G$, where $I := [0,1]$, which is homotopic to a constant path in R. Let $\tilde{\gamma} : I \to \pi^{-1}(G)$ be a lifting of γ to $\pi^{-1}(G)$ and E the connected component of $\pi^{-1}(G)$ which contains $\tilde{\gamma}(I)$. Then, we can verify that $\tilde{\gamma}$ is a closed path in E. Since G satisfies the strong disk property in R, the open set $\pi^{-1}(G)$ satisfies the strong disk property in Z. Then, by Proposition 2.3, E is Runge in Z. Since Z is Runge in $\mathbb{C}$, the open set E is also Runge in $\mathbb{C}$ and, therefore, E is simply connected. It follows that there exists a homotopy $\tilde{\eta}$ in E between $\tilde{\gamma}$ and a constant path. Then, $\pi \circ \tilde{\eta}$ is a homotopy in G between γ and a constant path. Thus, we proved that $\pi_1(G) \to \pi_1(R)$ is injective.

(2) → (1). The assertion is a direct consequence of Theorem 4.1 because every open set of an open Riemann surface R is meromorphically $\mathcal{O}(R)$-convex (see Abe [1, Proposition 16] or Abe [3, Theorem 5.2]). □

Remark 4.3. In the case where $\dim R \ge 2$, the converse of Theorem 4.1 is not true. Let, for example, $R := \mathbb{C}^2$ and $G := \{(z,w) \in \mathbb{C}^2 \mid |z| < 2, |w| < 2, |zw - 1| < 1/2\}$. Then, G is a Runge domain in $\mathbb{C}^2$ and, therefore, G is meromorphically $\mathcal{O}(\mathbb{C}^2)$-convex. On the other hand, G is not simply connected (see Nishino [14, p. 103]).

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