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On the convergence of approximate solutions of non-linear equation f(x)=0 for f(x) in C^1-class

C^1級関数$f(x)$に対する非線形方程式$f(x)=0$の近似解の収束性について

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Abstract

As the methods of finding the approximate solutions of non-linear equation $f(x) = 0$, the Newton and the secant methods are well known and are recently recognized again because they are fairly applicable to computer calculation.

In these iterative procedures, the curve $y = f(x)$ is approximated by the straight lines and hence the recurrence formulas are relatively simple and the successive calculations are not so difficult.

But, in general, the convergence of approximate sequence $\{x_n\}$ is not necessarily assured. Indeed, in many cases, the sequence $\{x_n\}$ does not approach to a true solution.

For giving the assurrance of the convergence of $\{x_n\}$ we must determine the suitable extent of starting point x_0.

The typical proof of the convergence of $\{x_n\}$ is done with the help of the so-called principle of contraction mapping under the additional condition that $f(x)$ is in the class C^2 .

The purpose of this paper is to prove the convergence of $\{x_n\}$ when $f(x)$ is in C^1 but not necessarily in C^2 .

And further, the author should like to remark that his method does not ask the help of the principle of contraction mapping and that his new proof is applicable to the case of the more complicated secant method whose approximate sequence is determined by recursion formula of three poins.

1. Newton method and known result

In the classical Newton method, the approximate solutions are obtained by replacing the curve $y = f(x)$ with the tangent lines and the recurrence relation is of the form

$$(\star) \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \ , \qquad n = 0,\ 1,\ 2,\ \cdots \ .$$

The above approximate sequence $\{x_n\}$ does not necessarily converge to a true solution.

For the convergence of $\{x_n\}$, we need some conditions on the smoothness of $f(x)$ and suitable choise of the starting point x_0 .

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The following theorem is well known.

Theorem A. *Let I be a closed interval containing the solution α of equation $f(x)=0$ in its interior. Suppose that $f(x)$ is in $C^2(I)$ and that $f'(x)\neq 0$ on I . Then there exists a subinterval $I'\subset I$ such that the approximate sequenc $\{x_n\}$ defined by $(\star)$ converges to α for any starting point $x_0\in I'$. And further, there exists a constant C satisfying*

$$|x_{n+1}-\alpha|\leq C|x_n-\alpha|^2 ,$$

for any $n\geq 0$.

Remark. When we investigate the above theorem A and its proof, we notice at once the following point at issue:

The most popular proof is done by using the principle of contraction mapping. So we need to differenciate $g(x)=x-\frac{f(x)}{f'(x)}$ for finding a Lipschitz constant L satisfying $|g'(x)|\leq L<1$ and therefore we must suppose that $f(x)$ is in the class C^2 . But this additional condition is unnatural because $f''(x)$ does't appear in the recursion function $g(x)$.

In the following sections, we shall discuss the problems concerning the point above mentioned.

2. Convergence of approximate sequence for the function $f(x)$ in C^1

First we have the following new

Theorem 1. *Let α be a true solution of $f(x)=0$ and I be a closed interval containing α in its interior. Suppose that $f(x)$ is in $C^1(I)$ and that $f'(x)\neq 0$ on I .*

Then there exists a subinterval $\tilde{I}$ of I such that the approximate sequence obtained by the Newton method converges to α for any starting point x_0 in $\tilde{I}$.

Proof. By the mean value theorem, there exists a point ξ in (α,x_0) or in (x_0,α) such that

$$\begin{aligned} x_1-\alpha=x_0-\alpha-\frac{f(x_0)}{f'(x_0)}&=x_0-\alpha-\frac{f(x_0)-f(\alpha)}{f'(x_0)}\\ &=x_0-\alpha-\frac{f'(\xi)}{f'(x_0)}(x_0-\alpha)=\frac{f'(x_0)-f'(\xi)}{f'(x_0)}(x_0-\alpha). \end{aligned}$$

Therefore we have

$$|x_1-\alpha|=|\frac{f'(x_0)-f'(\xi)}{f'(x_0)}|\,|x_0-\alpha|,$$

we remark here that $f'(x)\neq 0$ on I and that $f(\alpha)=0$.

Put $\delta = \min_{x\in I} |f'(x)| > 0$.
By our assumption that $f(x) \in C^1$, $f'(x)$ is uniformly continuous on I .
Then there exists, for $\forall k \in (0,1)$, a number $d > 0$ satisfying

$$|f'(y_1) - f'(y_2)| < k\delta \ , \qquad \forall y_i \in [\alpha - d, \alpha + d].$$

The interval $\tilde{I} = [\alpha - d, \alpha + d]$ is just the suitable extent of the starting point . Choosing α sufficiently small, we may suppose that $\tilde{I}$ is contained in I .
If x_0 is in $\tilde{I}$, we have

$$|x_1 - \alpha| = |\frac{f'(x_0) - f'(\xi)}{f'(x_0)}| \, |x_0 - \alpha| < \frac{k\delta}{\delta}|x_0 - \alpha| \leq k\delta < d$$

and hence x_1 is in $\tilde{I}$.
Repeating this procedure, we can prove that $\{x_n\}_{n=1}^{\infty} \in \tilde{I}$ and further we obtain

$$|x_n - \alpha| < k^n d.$$

The number k satisfying $0 < k < 1$, we have

$$\lim_{n\to\infty} x_n = \alpha \ .$$

This complete the proof.

Remark. (1) In our proof we need not the help of the principle of contraction mapping and so we can omit to differenciate the recurrence function $g(x)$. therefore, in the above new theorem, we don't suppose that $f(x)$ is in $C^2(I)$. It suffices to notice that $f'(x)$ is uniformly continuous on I .

(2) We can not also determine explicitely the suitable subinterval $\tilde{I}$ of I similarly in the classical proof depend on the principle of contraction mapping.

(3) The order of convergence of approximate sequence reduces to 1 when we don't suppose that $f(x)$ is in $C^2(I)$.

4. Convergence of approximate sequence by the secant method

The approximate solution of the secant method is the intersection point of a chord joining two points on the curve $y = f(x)$ and the x-axis. The recurrence formula of the secant method is concerned with the neighbouring 3 points and is defined by

$$(\star\star) \qquad x_{n+2} = x_{n+1} - \frac{f(x_{n+1})}{f(x_{n+1}) - f(x_n)}(x_{n+1} - x_n), \quad n = 0, 1, 2, \cdots .$$

The speed of convergence of approximate sequence obtained by the secant method is slower than that of the Newton method. The following theorem corresponding to Theorem A is well known.

Theorem B. *Let I be a closed interval containing a solution α of $f(x)=0$ in its interior. Suppose that $f(x)$ is in $C^2(I)$ and that $f'(x)\neq 0$ on I .*

Then there exists a subinterval I' of I such that the approximate sequence $\{x_n\}$ defined by recurrence formula ($\star\star$) converges to α for any x_0 and x_1 in I' . Further, there exists a constant C satisfying

$$|x_{n+2}-\alpha| \leq C|x_{n+1}-\alpha||x_n-\alpha| \quad for \ \ \forall n \geq 0 .$$

For the function $f(x)$ in C^1 but necessarily in C^2 , we can prove the following

Theorem 2. *Let α be a true solution of $f(x)=0$ and I be a closed interval containing α in its interior. Suppose that $f(x)$ is in $C^1(I)$ and that $f'(x)\neq 0$ on I .*

Then there exists a subinterval $\tilde{I}$ of I such that the approximate sequence obtained by the secant method converges to α for any starting points x_0 and x_1 in $\tilde{I}$.

Proof. By the recurrence formula ($\star\star$) and by the mean value theorem, there exist ξ and η in I such that

$$\begin{aligned} x_2-\alpha &= x_1-\alpha-\frac{x_1-x_0}{f(x_1)-f(x_0)}f(x_1) \\ &= x_1-\alpha-\frac{x_1-x_0}{f(x_1)-f(x_0)}\big(f(x_1)-f(\alpha)\big) \\ &= x_1-\alpha-\frac{x_1-x_0}{f'(\xi)(x_1-x_0)}f'(\eta)(x_1-\alpha) \\ &= (x_1-\alpha)\left(1-\frac{f'(\eta)}{f'(\xi)}\right), \end{aligned}$$

we remark that $f(\alpha)=0$ and that $f'(x)\neq 0$ on I .

Therefore we have

$$|x_2-\alpha| = |x_1-\alpha|\left|\frac{f'(\xi)-f'(\eta)}{f'(\xi)}\right|.$$

The assumption $f(x)$ in $C^1(I)$ asserts that $f'(x)$ is uniformly continuouson I . Put $\delta=\min_{x\in I}|f'(x)|>0$.

Then there exists, for $\forall k\in(0,1)$, a number $d>0$ satisfying

$$|f'(y_1)-f'(y_2)|<k\delta \quad , \quad \forall y_i\in[\alpha-d,\alpha+d].$$

The interval $\tilde{I}=[\alpha-d,\alpha+d]$ is just the suitable extent of the starting points x_0 and x_1 . Choosing d sufficiently small, we may assume that $\tilde{I}$ is contained in I .

If x_0 and x_1 are in $\tilde{I}$, ξ and η are in $\tilde{I}$,we have

$$|x_2-\alpha| = \left|\frac{f'(\xi))-f'(\eta)}{f'(\xi)}\right||x_1-\alpha| < \frac{k\delta}{\delta}|x_1-\alpha| \leq k\delta < d,$$

and hence x_2 is in $\tilde{I}$.

Repeating this procedure, we can prove that $\{x_n\}_{n=1}^{\infty} \in \tilde{I}$ and further we obtain

$$|x_n - \alpha| < k^{n-1}d.$$

The number k satisfying $0 < k < 1$, we have $\lim_{n\to\infty} x_n = \alpha$.
This complete the proof.

Remark. (1) We obtained Theorem 2 for a function $f(x)$ in C^1 not necessarily in C^2 .

(2) We can not also determine explicitely the subinterval $\tilde{I}$ of I .

(3) The order of covergence of approximate sequence $\{x_n\}_{n=1}^{\infty}$ deduces to 1 when we suppose that $f(x)$ is in C^1 and hence the speed of the convergence becomes slower than that of the approximate sequence in Theorem B .

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