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A note on symbolic powers of regular prime ideals

正則素イデアルの記号的べきについての一注意

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Abstract. Let k be a field of arbitrary characteristic and let R be a k-algebra of finite type. In this paper we shall show that for $p \in \operatorname{Reg}\operatorname{Spec} R$ suppose the residue class field K of R_p is separable extension of k, then $D^n(p) = p^{(n+1)}$ for all $n \geq 1$.

1. Preliminaries.

Throughout this paper, let us denote by k a field of arbitrary characteristic and let R be a k-algebra. By a k-higher derivation $\delta = \{\delta_q\}$ of finite rank n on R, we shall mean a finite sequence of endomorphisms $\delta_0, \delta_1, \cdots, \delta_n$ of R as a k-vector space which satisfy the following two properties:

(1) δ_0 is the identity map of R, and

(2) for every r $(0 \leq r \leq n)$, and for all $x, y \in R$, we have

$$\delta_r(xy) = \sum_{i+j=r} \delta_i(x)\,\delta_j(y)\,.$$

We shall denote the collection of all such k-higher derivations of finite rank n on R by $H_k^n(R)$. On the other hand let us denote by $\operatorname{Der}_k^n(R)$ the R-module of all n-th order k-derivations of R to R. Thus $\varphi \in \operatorname{Der}_k^n(R)$ if and only if $\varphi \in \operatorname{Hom}_k(R, R)$, and for all $x_0, x_1, \cdots, x_n \in R$ we have

$$\varphi(x_0 x_1 \cdots x_n) = \sum_{s=1}^{n} (-1)^{s-1} \sum_{i_1 < \cdots < i_s} x_{i_1} \cdots x_{i_s} \varphi(x_0 \cdots \hat{x}_{i_1} \cdots \hat{x}_{i_s} \cdots x_n)\,.$$

For every component δ_r of $\delta = \{\delta_r\} \in H_k^n(R)$, δ_r is an r-th order derivation of R. Let D^n denote the set of composites $\delta_{\alpha_1}^{(1)} \cdots \delta_{\alpha_q}^{(q)}$, where each $\delta_{\alpha_i}^{(i)}$ is a component of an element of $H_k^n(R)$, and $\alpha_1 + \cdots + \alpha_q \leq n$, q arbitrary. For an ideal I of R, define

$$D^n(I) = \{f \in I : \varphi(f) \in I \quad \text{for every}\, \varphi \in D^n\}\,.$$

Lemma 1 ([1] Proposition 1). *$D^n(I)$ is an ideal of R, and we have $I^{n+1} \subset D^n(I)$.*

Lemma 2 ([1] Proposition 2). *If Q is a primary ideal of R, then so is $D^n(Q)$.*

Consider now a localization $\lambda : R \longrightarrow S^{-1}R$ of R. For every ideal I of R let $S(I) = \lambda^{-1}(S^{-1}I)$ be the S-saturation of I. On the other hand, every higher derivation $\delta = \{\delta_r\}$ on R can be extended uniquely to $\bar{\delta}=\{\bar{\delta}_r\}$ on $S^{-1}R$. Then let $\bar{D}^n$ denote the set of composites $\bar{\delta}_{\alpha_1}^{(1)} \cdots \bar{\delta}_{\alpha_q}^{(q)}$, where each $\bar{\delta}_{\alpha_i}^{(i)}$ is a component of a unique extension to $S^{-1}R$ of an element in $H_k^n(R)$, and $\alpha_1 + \cdots + \alpha_q \leq n$. It is clear that we have

$$\bar{D}^n \subset \operatorname{Der}_k^n(S^{-1}R)\,,$$

where $\mathrm{Der}_k^n(S^{-1}R)$ is the set of all n-th order k-derivations of $S^{-1}R$ to $S^{-1}R$. For an ideal $\bar{I}$ of $S^{-1}R$, denote by $\bar{D}^n(\bar{I})$ the set of $f \in \bar{I}$ such that $\bar{\varphi}(f) \in \bar{I}$ for every $\bar{\varphi} \in \bar{D}^n$.

Lemma 3 ([1] Proposition 3). *$\bar{D}^n(S^{-1}I) = S^{-1}D^n(S(I))$. In particular, $\bar{D}^n(S^{-1}Q) = S^{-1}D^n(Q)$ for a primary ideal Q of R such that $Q \cap S = \phi$, the empty set.*

Lemma 4 ([1] Proposition 4). *Let $\lambda : R \longrightarrow S^{-1}R$ be a localization of R, and let Q be a primary ideal of R such that $Q \cap S = \phi$. Then*

$$D^n(Q) = \lambda^{-1}\bar{D}^n(S^{-1}Q).$$

2. Results.

Proposition 5. *Let R be a k-algebra of finitely generated type which is a regular local ring with the maximal ideal m. Let K be the residue class field of R. Assume that K is a separable extension of k. Then $D^n(m) = m^{n+1}$ for all $n \geq 1$.*

Proof. By Lemma 1 we have $m^{n+1} \subset D^n(m)$. We shall show the converse inclusion relation. Let $\{z_1, \cdots, z_r\}$ be a regular system of parameters for R. Consider $\hat{R}$, the m-adic completion of R. Then $\hat{R}$ is expressed as a formal power series ring $K[\![z_1, \cdots, z_r]\!]$. Let

$$\delta^{(i)} = \{\delta_j^{(i)}\}_{j \leq n} \in H_K^n(\hat{R})\,, \qquad 1 \leq i \leq r$$

be the higher derivation defined by

$$\delta_j^{(i)}(z_1^{m_1} \cdots z_i^{m_i} \cdots z_r^{m_r}) = \binom{m_i}{j} z_1{}^{m_1} \cdots z_i^{m_i - j} \cdots z_r^{m_r}\,,$$

where we put $\binom{m_i}{j} = 0$ for $j > m_i$. With $\hat{m} = (z_1, \cdots, z_r)\hat{R}$ we have: If $f \in \hat{m}$ and if $\delta_{j_1}^{(1)} \cdots \delta_{j_r}^{(r)}(f) \in \hat{m}$ for all $j_1, \cdots, j_r$ such that $j_1 + \cdots + j_r \leq n$, then $f \in \hat{m}^{n+1}$. For, let $f = F_n + g$, where $F_n \in K[z_1, \cdots, z_r]$ is a polynomial of degree n and $g \in \hat{m}^{n+1}$. Then it is easily seen that $F_n = 0$. Assume that we have already exhibited $\partial^{(i)} = \{\partial_j^{(i)}\}_{j \leq n} \in H_k^n(R)$ such that $\partial_j^{(i)} = \delta_j^{(i)}|R$ for all i, j. Then we obtain what we want : Let $f \in m$ be such that $\varphi(f) \in m$ for every $\varphi \in D^n$. In particular, we have

$$\partial_{j_1}^{(1)} \cdots \partial_{j_r}^{(r)}(f) \in m \quad \text{for all} \quad j_1, \cdots, j_r$$

with $j_1 + \cdots + j_r \leq n$, hence

$$\delta_{j_1}^{(1)} \cdots \delta_{j_r}^{(r)}(f) \in \hat{m} \quad \text{for all} \quad j_1, \cdots, j_r$$

with $j_1 + \cdots + j_r \leq n$. This implies $F_n = 0$ and thus

$$f = g \in \hat{m}^{n+1} \cap R = m^{n+1}$$

It remains to show that there exist $\partial^{(i)} = \{\partial_j^{(i)}\} \in H_k^n(R)$, $1 \leq i \leq r$, such that $\partial_j^{(i)} = \delta_j^{(i)}|R$ for every i, j. Let $\Omega_k(R)$ be the universal algebra of higher differentials on R over k and let

$$\delta = \{\delta_j\} : R \longrightarrow \Omega_k(R)$$

be the canonical k-higher derivation of infinite rank (Cf.[2]). Since K is a separable extention of k, we can choose $u_1, \cdots, u_s \in R$ such that their images in K form a separating transcendence base of K over k. Then $\Omega_k(R)$ is a free R-algebra with a free base

$$\{\delta_j(z_l), \delta_j(u_m) : l = 1, \cdots, r, m = 1, \cdots, s, j = 1, 2, \cdots, \infty\}$$

([2] Theorem 3). On the other hand it is easily shown that each $\delta^{(i)} = \{\delta_j^{(i)}\}_{j\leq n} \in H_K^n(\hat{R})$ can be imbedded into a higher derivation $\{\delta_j^{(i)}\}$ of infinite rank. Hence there are uniquely determined k-higher derivations $\partial^{(i)} = \{\partial_j^{(i)}\}$ on R of infinite rank such that

$$\partial_j^{(i)}(z_l) = \delta_j^{(i)}(z_l), \quad \partial_j^{(i)}(u_m) = 0$$

for all i, j, l, m. Consequently $\partial_j^{(i)}=\delta_j^{(i)}|R$ for all i, j. Hence $\{\partial_j^{(i)}\}_{j\leq n} \in H_k^n(R)$, $1 \leq i \leq r$, are required ones.

Theorem 6. *Let k be a field of arbitrary characteristic and let R be a k-algebra of finite type. For $p \in \operatorname{Reg}\operatorname{Spec} R$ suppose the residue class field K of R_p is separable extension of k, then $D^n(p) = p^{(n+1)}$ for all $n \geq 1$.*

Proof. Let $\lambda : R \longrightarrow R_p$ be the canonical homomorphism and set $m = pR_p$. Then by Lemma 4, we have

$$D^n(p) = \lambda^{-1}\bar{D}^n(m).$$

Let $\{\delta_j\}_{j\leq n}$ be a k-higher derivation of R_p of rank n. Then there exist elements $t_i \in R - p$, i=1,$\cdots$, n, such that $\{\delta_0, t_1\delta_1, \cdots, t_n\delta_n\}$ is a k-higher derivation of rank n on R ([3], Lemma 2). Let us set

$$\partial_i = t_i\delta_i, \quad i = 0, 1, \cdots, n, \quad t_0 = 1.$$

Denoting by $\{\bar{\partial}_i\}_{i\leq n}$ the unique extension of $\{\partial_i\}_{i\leq n}$ to R_p, we have $\delta_i = (1/t_i)\bar{\partial}_i$ on R_p, $i = 0, 1, \cdots, n$. Let

$$\varphi = \delta_{\alpha_1}^{(1)} \cdots \delta_{\alpha_q}^{(q)}$$

be a composite of components of higher derivations on R_p. Then there exist elements $t_i \in R-p$, $i = 1, \cdots, q$, and a family of higher derivations $\{\partial_j^{(i)}\}$, $i = 1, \cdots, q$, on R such that

$$\varphi = (\frac{1}{t_1}\bar{\partial}_{\alpha_1}^{(1)}) \cdots (\frac{1}{t_q}\bar{\partial}_{\alpha_q}^{(q)}).$$

Here we denote by $\bar{\partial}_{\alpha_i}^{(i)}$ the unique extension of $\partial_{\alpha_i}^{(i)}$ to R_p. It is obvious that φ is an R_p-linear combination of elements of $\bar{D}^n$ and consequently $\bar{D}^n(m)$=m^{n+1} by Lemma 5. Therefore we have

$$D^n(p) = \lambda^{-1}(m^{n+1}) = p^{(n+1)} \quad \text{for all } n \geq 1.$$

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