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On the Mandelbrot Set of $w=z^n+c$

$w=z^n+c$ のMandelbrot集合について

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The Mandelbrot set of $w=z^2+c$ is well-known. The purpose of this paper is to investigate the properties of the Mandelbrot set M_n of $w=z^n+c$ for $n=3, 4, 5, \cdots$. As a consequence, we shall see that the shape of M_n is very close to the closed unit disc for sufficiently large n. Further, we shall give the explicit formulas for the 2-cycles of M_n and the 3-cycles of M_2.

§1. **The Mandelbrot set M_n.** Let $w=P_{c,n}(z)=z^n+c$ be a complex-valued function of a complex variable z with some complex constant c and with some integer $n \geqq 2$. We consider the iteration of $w=P_{c,n}(z)$ and denote the k-th iterate of $w=P_{c,n}(z)$ by $w=P_{c,n}{}^k(z)$. The Mandelbrot set M_n of $w=P_{c,n}(z)$ is the set of values of c's for which the sequence $\{P_{c,n}{}^k(0)\}$ $(k=1, 2, 3, \cdots)$ is bounded, that is, $M_n=\{c \mid |P_{c,n}{}^k(0)| \leqq A_{c,n}, k=1, 2, 3, \cdots\}$, where $A_{c,n}$ is a constant depending on c and n.

Theorem 1. Setting $L_{n,k}=\{c \mid |P_{c,n}{}^k(0)| \leqq \sqrt[n-1]{2}\}$, then, $L_{n,1} \supset L_{n,2} \supset L_{n,3} \supset \cdots$, and we have $M_n=\bigcap_{k=1}^{\infty} L_{n,k}$.

Proof. First, suppose $|P_{c,n}(0)|=|c|>\sqrt[n-1]{2}$, then, we have

$$|P_{c,n}{}^2(0)|=|c^n+c| \geqq |c|^n-|c| \geqq |c|(|c|^{n-1}-1),$$

and by induction,

$$|P_{c,n}{}^k(0)| \geqq |c|(|c|^{n-1}-1)^{n^{k-2}+\cdots+n+1}.$$

From these inequalities, we see that if $c \notin L_{n,1}$, then $c \notin L_{n,2}, L_{n,3}, \cdots$ as $|P_{c,n}{}^k(0)|>\sqrt[n-1]{2}$ for $k=2, 3, \cdots$ and $c \notin M_n$ as $|P_{c,n}{}^k(0)| \to \infty$ $(k \to \infty)$. Therefore, we have $L_{n,1} \supset L_{n,2}, L_{n,3}, \cdots$ and $L_{n,1} \supset M_n$.

Next, suppose $|P_{c,n}{}^k(0)|>\sqrt[n-1]{2}$, that is, $c \notin L_{n,k}$. If $|c|>\sqrt[n-1]{2}$, we have already seen that $c \notin L_{n,k+1}, L_{n,k+2}, \cdots$ and $c \notin M_n$. If $|c| \leqq \sqrt[n-1]{2}$, setting $|P_{c,n}{}^k(0)|=h$, we have

$$|P_{c,n}{}^{k+1}(0)|=|\{P_{c,n}{}^k(0)\}^n+c| \geqq |P_{c,n}{}^k(0)|^n-|c| \geqq h^n-h \geqq h(h^{n-1}-1),$$

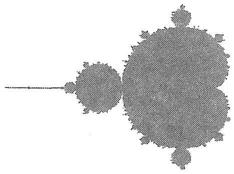
and in a similar way,

$$|P_{c,n}{}^{k+s}(0)| \geqq h(h^{n-1}-1)^{n^{s-1}+\cdots+n+1},$$

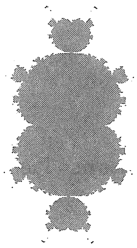
This implies that if $c \notin L_{n,k}$, then $c \notin L_{n,k+1}, L_{n,k+2}, \cdots$ as $|P_{c,n}{}^{k+s}(0)|>\sqrt[n-1]{2}$ for $s=1, 2, \cdots$ and $c \notin M_n$ as $|P_{c,n}{}^{k+s}(0)| \to \infty$ $(s \to \infty)$. Therefore, we have $L_{n,k} \supset L_{n,k+1}, L_{n,k+2}, \cdots$ and $L_{n,k} \supset M_n$.

Hence, we have $L_{n,1} \supset L_{n,2} \supset L_{n,3} \supset \cdots$ and $M_n \subset \bigcap_{k=1}^{\infty} L_{n,k}$. As $M_n \supset \bigcap_{k=1}^{\infty} L_{n,k}$ is trivial, we obtain the theorem. Q.E.D.

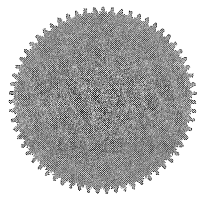
According to Theorem 1, M_n is the intersection of the closed subsets of the complex plain C bounded by $\sqrt[n-1]{2}$, so that, M_n is also the closed subset of C bounded by $\sqrt[n-1]{2}$. Further, if $C - M_n$ contains a bounded component, some $C - L_{n,k}$ also contains a bounded component, which is a contradiction by the maximum principle. Therefore, $C - M_n$ contains no bounded components and M_n is simply connected. Later, we shall see that M_n consists of only one component. We shall give the pictures of M_2, M_3 and M_{64} by using the computor graphics.



M_2



M_3



M_{64}

§2. The main component of the interior of M_n. The interior of the Mandelbrot set M_n consists of infinitely many components. Among these components, we denote the largest one containing $c = 0$ by W_n.

On the other hand, we consider the set of those c's for which the fixed point of $w = z^n + c$ has its multiplier λ satisfying $|\lambda| < 1$. We call this set the attracting cycles of $w = z^n + c$ and denote it by $D_{n,1}$. Let α be the fixed point, then, we have $\alpha^n + c = \alpha$ and $n\alpha^{n-1} = \lambda$ with $|\lambda| < 1$. Therefore, we have $D_{n,1} = \{c \mid c = \alpha - \alpha^n,\ |\alpha| < \frac{1}{\sqrt[n-1]{n}}\}$.

Theorem 2. $D_{n,1} = W_n$.

Proof. First, we shall prove $D_{n,1} \subset M_n$. Let c be the point of $D_{n,1}$. If $c \notin M_n$, then, we have $|P_{c,n}{}^k(0)| \to \infty$ $(k \to \infty)$ for the critical point z=0 of $w=z^n+c$. Therefore, the Julia set of $w=z^n+c$ is totally disconnected. On the other hand, the fixed point of $w = z^n+c$ is attracting, so that the Julia set of $w = z^n+c$ is the boundary of the basin of attraction around the fixed point. This is a contradiction, and we have $c \in M_n$.

Next, we shall prove $D_{n,1} = W_n$. $D_{n,1} \subset W_n$ is trivial. Now, suppose there exists a point c of $W_n - \overline{D}_{n,1}$, where $\overline{D}_{n,1}$ is the closure of $D_{n,1}$ in C. Then, as $c \in W_n \subset M_n$, the sequence $\{P_{c,n}{}^k(0)\}$ $(k=1,2,3,\cdots)$, which are the functions of c, is uniformly bounded, so that, the subsequence of $\{P_{c,n}{}^k(0)\}$ converges uniformly to the function $\phi_n(c)$ which is the fixed point of $w = z^n + c$. On the other hand, as $c \notin \overline{D}_{n,1}$, this fixed point is repelling. Therefore, we have $P_{c,n}{}^k(0) = \phi_n(c)$ for some sufficiently large k. The values of c's satisfying these equations are countable. This is a contrdiction, and we have $D_{n,1} = W_n$. Q.E.D.

According to Theorem 2, we see that the sequence of functions $\{P_{c,n}{}^k(0)\}$ of c converges in $D_{n,1}$ to the function $\phi_n(c)$ which is the branch of the algebraic function $\phi_n(c)^n - \phi_n(c) + c = 0$ satisfying $\phi_n(0) = 0$.

Further, combining Theorem 1 and Theorem 2, we obtain the following theorem.

Theorem 3. The boundary of M_n is contained in the closed set $\{c \mid \frac{1}{\sqrt[n-1]{n}}(1-\frac{1}{n}) \leqq c \leqq \sqrt[n-1]{2}\}$, for which the equalites $\lim_{n\to\infty} \frac{1}{\sqrt[n-1]{n}}(1-\frac{1}{n})=1$ and $\lim_{n\to\infty} \sqrt[n-1]{2}=1$ are satisfied.

§3. The attracting cycles of M_n. We further investigate the set of values of c's for which the fixed point of $w=P_{c,n}{}^k(z)$ has its multiplier λ satisfying $|\lambda|<1$. We call this set the attracting k-cycles of $w=z^n+c$ and denote it by $D_{n,k}$. Let α be the fixed point, then, we have $P_{c,n}{}^k(\alpha)=\alpha$ and $n^k\{P_{c,n}{}^{k-1}(\alpha)\cdots P_{c,n}(\alpha)\alpha\}^{n-1}=\lambda$ with $|\lambda|<1$. But, it is difficult to give the explicit formula for $D_{n,k}$ from these equations. Here, we shall give the explicit formulas for $D_{n,k}$ in the case of k=2 and in the case of k=3 and n=2.

In the case of k=2, instead of these equations, we consider the following equations.

$$\alpha^n+c=\beta,\quad \beta^n+c=\alpha,\quad \alpha\neq\beta,\quad n^2\alpha^{n-1}\beta^{n-1}=\lambda,\quad |\lambda|<1.$$

From these equations, setting $\alpha+\beta=\xi$, $\alpha\beta=\eta$, we have the following explicit formula for $D_{n,2}$.

Theorem 4. $D_{n,2}=\{c \mid \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \eta \\ -1 & \xi \end{pmatrix}^{n-2} \begin{pmatrix} \eta \\ \xi \end{pmatrix} + \begin{pmatrix} \xi \\ 1 \end{pmatrix},\ |\eta|<\frac{1}{\sqrt[n-1]{n^2}}\}$.

According to this formula, we have

$$D_{2,2}=\{c \mid c=\eta+\xi,\ \xi+1=0,\ |\eta|<\frac{1}{4}\}=\{c \mid |c+1|<\frac{1}{4}\},$$

which is the open disc of radius $\frac{1}{4}$ with its center c=-1. And also by this formula, we have

$$D_{3,2}=\{c \mid c=\eta\xi+\xi,\ \xi^2-\eta+1=0,\ |\eta|<\frac{1}{3}\}=\{c \mid c=\sqrt{\eta-1}(\eta+1),\ |\eta|<\frac{1}{3}\},$$

which is the domain inside the curve $c=\frac{1}{3\sqrt{3}}\sqrt{e^{i\theta/2}-3}\,(e^{i\theta/2}+3)$.

We remark that the sequence of functions $\{P_{c,n}{}^{2k}(0)\}$ $(k=1,2,3,\cdots)$ of c converges in $D_{n,2}$ to the function $\phi_n(c)$ which is the branch of the algebraic function $\{\phi_n(c)^n+c\}^n-\phi_n(c)+c=0$ satisfying $\phi_n(0)=0$, and that the sequence of functions $\{P_{c,n}{}^{2k+1}(0)\}$ $(k=1,2,3,\cdots)$ of c converges in $D_{n,2}$ to the function $\phi_n(c)^n+c$.

In the case of k=3, the equations are far complicated. We can give the explicit formula only for $D_{2,3}$. In this case, we consider the following equations.

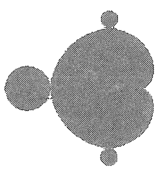
$$\alpha^2+c=\beta,\quad \beta^2+c=\gamma,\quad \gamma^2+c=\alpha,\quad \alpha\neq\beta,\quad 8\alpha\beta\gamma=\lambda,\quad |\lambda|<1.$$

From these equations, setting $\alpha\beta\gamma=\eta$, we have the following explicit formula for $D_{2,3}$.

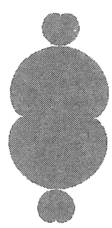
$$D_{2,3}=\{c \mid c^3+2c^2+(1-\eta)c+(1-\eta)^2=0,\ |\eta|<\frac{1}{8}\}.$$

This is the domain consisting of three components of the interior of the Mandelbrot set M_2. One is located on the main antenna of M_2, and the other two are tangent to the main component W_2 at $c=-\frac{1}{8}\pm\frac{3\sqrt{3}}{8}i$. These latter two components are the image of the algebraic function c of η, so that, they are not the open discs.

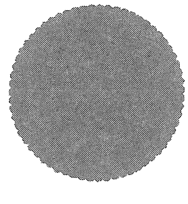
We shall give the pictures of $D_{2,1}\cup D_{2,2}\cup D_{2,3}$, $D_{3,1}\cup D_{3,2}$ and $D_{64,1}$ in the following.



$D_{2,1}\cup D_{2,2}\cup D_{2,3}$



$D_{3,1}\cup D_{3,2}$



$D_{64,1}$

According to [2], each component of $D_{2,k}$ is conformally equivalent to the open unit disc. It is not known that all the components of the interior of M_2 coinside with all the cycles.

§4. The Green's function of $\hat{C}-M_n$. We consider the function $\phi_{n,k}(c)=\sqrt[n^k]{P_{c,n}{}^{k+1}(0)}$. This is a well-defined, single-valued analytic function in $C-L_{n,k}$ and maps $C-L_{n,k}$ conformally onto $C-\{c \mid |c| \leqq \sqrt[n^k(n-1)]{2}\}$. Therefore, the limitting function $\phi_n(c)=\lim_{k\to\infty}\sqrt[n^k]{P_{c,n}{}^{k+1}(0)}$ maps $C-M_n$ conformally onto $C-\{c \mid |c| \leqq 1\}$. This shows $\hat{C}-M_n$, where $\hat{C}$ is the extended complex plain, is symply connected and we see that M_n is connected.

According to the above consideration, the Green's function of $\hat{C}-M_n$ with its pole at ∞ is given by $G_n(c,\infty,\hat{C}-M_n)=\log|\phi_n(c)|=\lim_{k\to\infty} n^{-k}\log|P_{c,n}{}^{k+1}(0)|$. Here, we can rewrite $\phi_n(c)$ as $\phi_n(c)=c\prod_{k=1}^{\infty}\left(1+\frac{c}{P_{c,n}{}^{k}(0)}\right)^{\frac{1}{n^k}}$, so that, we have $G_n(c,\infty,\hat{C}-M_n)=\log|\phi_n(c)|=\log|c|+o(1)$. Therefore, the Robin constant of M_n is equal to 0, and the logarithmic capacity of M_n is equal to 1. Considering the results in §1, we obtain the following theorem concerning M_n.

Theorem 5. The Mandelbrot set M_n of $w=z^n+c$ is a connected and simply connected clsed set in C bounded by $\sqrt[n-1]{2}$ with its logarithmic capacity equal to 1.

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