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一般化された標本化定理と近似的な標本化函数

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Generalized Sampling Theorem and Approximate Sampling Function

by
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Abstract

Someya-Shannon's sampling theorem[1)2)] is generalized so as to include sampled values and sampled derivatives. The sampling function can be chosen from many kinds of continuous functions, which are very similar to the delta-function with narrow breadth and low feet at both sides of the main peak. Several examples of the sampling functions are given. For an approximation of the sampling formulae, a proposal is made to use other kinds of sampling functions of character very similar to the delta function.

§ 1. Preliminaries

A generalized sampling theorem was presented by one of the authors, Takizawa[24)], which can be conveniently applied to construct the generakized interpolation formulae[3)~14)] in the fields of physics[15)16)] and engineering.

The present authors wish to discuss the structure of the generalized sampling theorem[17)~26)] and to make some comments to the generalized sampling functions. The sampling functions used here are very similar to the delta-function with narrow breadth at both sides of the main peak.

In practical application, one can make use of such δ-function-like continuous functions. The detailed examples of the approximate sampling functions shall be proposed in § 7.

§ 2. Generalized Sampling Theorem

At first, we shall write the generalized sampling theorem[24)]. It reads:

Theorem I (Generalized Sampling Theorem)[17)~25)28)]

An entire function f(Z) can be expressed by:

$$f(z)=\sum_{n}\sum_{k=0}^{m_n}\sum_{j=0}^{m_n-k}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(k)}}{k!}\cdot(z-z_n)^{j+k}\cdot\frac{g(z)}{(z-z_n)^{m_n+1}}$$

$$=\sum_{n}\sum_{s=0}^{m_n}\sum_{j=0}^{s}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot(z-z_n)^{s}\cdot\frac{g(z)}{(z-z_n)^{m_n+1}}$$

$$=\sum_{n}\sum_{s=0}^{m_n}\frac{(z-z_n)^s}{s!}\cdot\Big[\frac{d^s}{dz^s}\{f(z)\cdot H(z,z_n)\}\Big]_{z=z_n}\cdot\frac{g(z)}{(z-z_n)^{m_n+1}}\quad ,$$

(2 − 1)

where the series in the right-hand side of (2-1) converges uniformly in any bounded closed domain in the complex z-plane, if the following conditions are satisfied:

(I) f(z) and g(z) are entire, (2 − 2)

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(II) g(z) has zeros of (m_n+1)-th order at point $z=z_n$ (n=integer), i.e.

$$g(z_n)=g'(z_n)=g''(z_n)=\ldots=g^{(m_n)}(z_n)=0\,, \quad \text{and} \quad g^{(m_n+1)}(z_n)\neq 0\,, \tag{2-3}$$

for m_n non-negative integer, which depands on z_n, and

(III) $$\lim_{z\to\infty}\frac{f(z)}{g(z)}=0 \quad . \tag{2-4}$$

Here, for the sake of brevity, we write:

$$f_n^{(k)}=\left[\frac{d^k}{dz^k}f(z)\right]_{z=z_n}\,, \quad (k=0,1,2,\ldots)$$: sampled values and sampled derivatives

$$g_n^{(k)}=\left[\frac{d^k}{dz^k}g(z)\right]_{z=z_n}\,, \quad (k=0,1,2,\ldots)$$

$$H_n^{(k)}=\left[\frac{d^k}{dz^k}H(z,z_n)\right]_{z=z_n}=\frac{(-1)^k}{h_n}\begin{vmatrix} \frac{h'_n}{h_n}, & 1. & 0 & & \\ \frac{h''_n}{h_n}, & {}_2C_1\frac{h'_n}{h_n}, & 1. & & 0 \\ \vdots & \vdots & \vdots & & \\ \frac{h_n^{(k-1)}}{h_n}, & {}_{k-1}C_1\frac{h_n^{(k-2)}}{h_n}, & {}_{k-1}C_2\frac{h_n^{(k-3)}}{h_n}, & \ldots, & 1 \\ \frac{h_n^{(k)}}{h_n}, & {}_kC_1\frac{h_n^{(k-1)}}{h_n}, & {}_kC_2\frac{h_n^{(k-2)}}{h_n}, & \ldots, & {}_kC_{k-1}\frac{h'_n}{h_n} \end{vmatrix},$$

and

$$h_n^{(k)}=\left[\frac{d^k}{dz^k}h(z,z_n)\right]_{z=z_n} \cdot \quad (k=0,1,2,\ldots)$$

The summation in (2-1) is taken over all the points $z=z_n$ (n=integer). The function

$$h(z,z_n)\equiv\frac{1}{H(z,z_n)}\equiv\frac{g(z)}{(z-z_n)^{m_n+1}} \quad , \tag{2-6}$$

is called as a **generalized sampling function**, and points $z=z_n$ as **sampling points**. The expression (2-1) shall be called as a **generalized sampling formula**.

The proof of theorem I is straightforward. Under the conditions (I) and (II), the function f(z)/g(z) is meromorphic in the complex z-plane. It has poles of (m_n+1)-th orderat points $z=z_n$.By means of the Cauchy theorem, the function f(z)/g(z) can be expressed by a contour-integration along a circle of radius R with center at the origin, including poles of f(z)/g(z) in the circle $|z|=R$. If one takes the radius R to be infinitely large, then the contour-integration vanishes under the condition (III), and one has merely to calculate residues at points $z=z_n$. After calculating the summe of the residues, one multiplies g(z) to both sides of the expression thus obtained, and proves theorem I. (cf. Fig.1.)

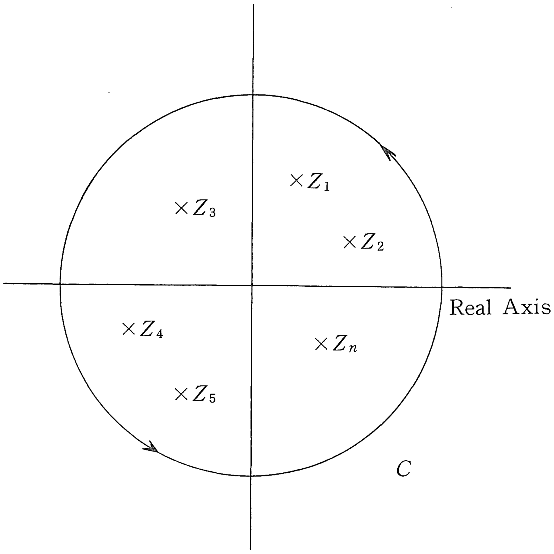

Fig.1. Poles of f(z)/g(z) and integration contour C

§ 3. Special Cases of Theorem I

If all the m_n are the same, one writes m instead of m_n, and one obtains the following Theorem II.

Theorem II Under the conditions (I), (II), and (III), we have:

$$f(z)=\sum_{n}\sum_{S=0}^{m}\sum_{j=0}^{S}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(k)}}{(s-j)!}\cdot(z-z_n)^S\cdot\frac{g(z)}{(z-z_n)^{m+1}}, \qquad (3-1)$$

for m which denotes the same value of all the m_n.

From Theorem II we obtain the following:

Theorem III If an entire function g(z) satisfies the conditions (I)~(III) and g(z) can be expanded into the Taylor series as follows:

$$g(z)=A_{m+1}\cdot(z-z_n)^{m+1}+\sum_{S=1}^{\infty}A_{2m+1+S}\cdot(z-z_n)^{2m+1+S}, \quad (A_{m+1}\neq 0)$$

then expression (3-1) is reduced to:

$$f(z)=\sum_{n}\sum_{S=0}^{m}f_n^{(j)}\cdot\frac{(z-z_n)^S}{s!}\cdot\frac{1}{\frac{g_n^{(m+1)}}{(m+1)!}}\cdot\frac{g(z)}{(z-z_n)^{m+1}} \cdot \qquad (3-2)$$

§ 4. Sampling Formula for small m

a) If all the poles of f(z)/g(z) are simple poles (i.e. m=0) at all the sampling points z_n, then expression (2-1) is simplified and we obtain a sampling formula[26]:

$$f(z)=\sum_{n}f_n\cdot\frac{g(z)}{(z-z_n)\cdot g'_n} \cdot \qquad (4-1)$$

b) If all the poles of f(z)/g(z) are of 2nd order (i.e. m=1) at all z_n, then expression (2-1) leads to a sampling formula containing sampled values f_n and sampled derivatives f'_n of first order:

$$f(z)=\sum_n \left[f_n+(z-z_n)\cdot\{f'_n-\frac{1}{3}f_n\cdot\frac{g_n^{(3)}}{g_n^{(3)}}\}\right]\cdot\frac{2!\cdot g(z)}{(z-z_n)^2\cdot g_n^{(2)}} \quad . \qquad (4-2)$$

c) If all the poles of f(z)/g(z) are of 3rd order (i.e. m=2) at points z_n, then we obtain sampling formula taking sampled heights f_n and sampled derivatives f'_n and f''_n into account:

$$f(z)=\sum_n\Big[f_n+(z-z_n)\cdot\{f'_n-\frac{1}{4}f_n\cdot\frac{g_n^{(4)}}{g_n^{(3)}}\}+\frac{1}{2}(z-z_n)^2\cdot\{f''_n-\frac{1}{2}f'_n\cdot\frac{g_n^{(4)}}{g_n^{(3)}}+$$

$$+\frac{1}{2}f_n\cdot[\frac{1}{4}(\frac{g_n^{(4)}}{g_n^{(3)}})^2-\frac{1}{5}\frac{g_n^{(5)}}{g_n^{(3)}}]\}\Big]\cdot\frac{3!\cdot g(z)}{(z-z_n)^3\cdot g_n^{(3)}} \quad . \qquad (4-3)$$

d) In case of m>0, it is practically convenient to take

$$g(z)=\psi^{m+1}(z), \qquad (4-4)$$

where $\psi(z)$ is an entire function which has merely simple zeros at all the sampling points $z=z_n$. So, the sampling formula is expressed as follows:

$$f(z)=\sum_n\sum_{s=0}^{m}\sum_{j=0}^{s}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot(z-z_n)^s\cdot\frac{\psi^{m+1}(z)}{(z-z_n)^{m+1}} \quad , \qquad (4-5)$$

where $H_n^{(s)}$'s are given in (2-5), with

$$h_n=g_n^{(m+1)}/(m+1)!=(\psi'_n)^{m+1}, \qquad (4-6)$$

and

$$h_n^{(r)}=\frac{r!}{(m+1+r)!}\cdot g_n^{(m+1+r)}$$

$$=\sum_{\substack{p+q+u+\dots=m+1\\ p+2q+3u+\dots=m+1+r}}\frac{r!\cdot(m+1)!}{p!q!u!\dots}\cdot(\psi'_n)^p\cdot(\frac{1}{2!}\psi''_n)^q\cdot(\frac{1}{3!}\psi'''_n)^u\dots \quad . \qquad (4-7)$$

Here the present authors want to emphasize that the sampling formulae above mentioned can be conveniently applied as **interpolation formulae**, while these formulae are not very useful as extrapolation formulae, when one truncates the sampling expansion. Because individual term in the series plays equally a rôle and one can not simply ignore certain number of terms in the expansion.

§ 5. Detailed Examples of the Generalized Sampling Formula for m=0

For m=0, formula (4-1) can be applied.

a) One takes an orthogonal system of polynomials in the domain $a\leqq z\leqq b$:

$$\{\phi_k(z)|k=1,2,3,\dots,\ \int_a^b\omega(z)\phi_m(z)\phi_n(z)dz=\delta_{m,n}\} \quad , \qquad (5-1)$$

with a polynomial $\phi_s(z)$ of s-th degree, and density fumction $\omega(z)$. From (4-1) and taking g(z) as $\phi_s(z)$, one can obtain an approximation formula $\bar{f}(z)$ for f(z):

$$\bar{f}(z)=\sum_{n=1}f(z_n)\cdot\frac{\phi_s(z)}{(z-z_n)\cdot\phi'_s(z_n)} \quad . \qquad (5-2)$$

This formula relates to Gauß' quadratur formula and Christoffel's number, if we consider an integral[19]:

$$\int_a^b \omega(z)\bar{f}(z)\mathrm{d}z \quad .$$

b) In order to obtain Lagrangean formula, one takes

$$g(z)=\prod_{k=1}^{s}(z-z_k), \quad , \qquad (5-3)$$

and obtain an approximate formula $\bar{f}(z)$:

$$\bar{f}(z)=\sum_{n=1}^{s}f(z_n)\cdot L_n(z) \quad , \qquad (5-4)$$

with

$$L_n(z)=\prod_{k=1,k\neq n}^{s}\frac{z-z_k}{z_n-z_k} \quad , \qquad (5-5)$$

and

$L_n(z_p)=\delta_{n,p}$. $(1\leqq p\leqq s)$ (5 − 6)

c) One takes a Chebyshev's polynomial:

$g(z)=\cos(\alpha \arccos \beta z)$, ($\alpha$=positive integer and $\beta\neq 0$) (5 − 7)

and obtain an approximate formula $\bar{f}(z)$:

$$\bar{f}(z)=\frac{1}{\alpha}\sum_{n}(-1)^n\cdot f(z_n)\cdot\sqrt{1-\beta^2 z_n^2}\cdot\frac{\cos(\alpha \arccos \beta z)}{\beta(z-z_n)} \qquad (5-8)$$

with

$\cos(\alpha \arccos \beta z_n)=0$,, (5 − 9)

i.e.

$\beta z_n=\cos\{(2n+1)\pi/2\alpha\}$. (n=integer) (5 −10)

d) One takes

$g(z)=\sin(\alpha z+\beta)$, ($\alpha$, β=const, $\alpha\neq 0$) (5 −11)

and obtain Someya-Shannon's sampling formula

$$f(z)=\sum_{n=-\infty}^{+\infty}f\left(\frac{n\pi-\beta}{\alpha}\right)\cdot\frac{\sin(\alpha z+\beta-n\pi)}{\alpha z+\beta-n\pi} \quad . \quad (\alpha\neq 0) \qquad (5-12)$$

Shannon's formula[1)2)] corresponds to (5-12) when $\alpha=1$ and $\beta=0$.

e) As for other examples, we can take:

i) $g(z)=z\sin(\alpha z)$, $(\alpha\neq 0)$ (5 −13)

ii) $g(z)=\alpha z\sin(\alpha z)-A\cos(\alpha z)$, $(\alpha A\neq 0)$ (5 −14)

iii) $g(z)=\alpha z\cos(\alpha z)-B\sin(\alpha z)$, $(\alpha\beta\neq 0)$ (5 −15)

iv) $g(z)=J_\nu(\alpha z)$, (ν=integers, $\alpha\neq 0$) (5 −16)

v) $g(z)=\alpha z J'_\nu(\alpha z)+hJ_\nu(\alpha z)$, ($\nu$=integers, $\alpha h\neq 0$) (5 −17)

vi) $g(z)=T_\nu(\alpha z, \beta z)$, ($\nu$=integers, $\alpha\beta\neq 0$) (5 −18)

with

$T_\nu(x, y)=N_\nu(x)J_\nu(y)-J_\nu(x)N_\nu(y)$, ($\nu$=integers, $x>0$, and $y>0$)

where $J_\nu(z)$ and $N_\nu(z)$ are Bessel and Neumann functions, respectively.

An example of the formula for $g(z)=J_1(\alpha z)$ is given by Wheelon[27].

vii) Mathieu functions: $ce_n(z)$ and $se_n(z)$, (5 −19)

viii) Inverted Gamma function, $1/\Gamma(z)$, (5 −20)

etc.

§ 6. Sampling Formula for $m \geqq 1$

For the case $m=1$ or $m=2$, we can take in (4-4) squared functions or cubic functions of g(z) expressed in (5-1), (5-3), (5-7), (5-11), and (5-13) ~ (5-20), and obtain sampling formulae including sampled function and sampled higher order derivatives by means of (4-2) or (4-3). The detailed expression of the sampling formulae we shall omit here.

§ 7.Approximate Sampling Formula

If we restrict ourselves to the case $m=0$ and want to have some approximate sampling formulae, we may use a sampling function, which is very similar to the delta function having narrow breadth and low feet at both sides of the main peak. This idea is essentially based on the sampling theorems cited above, but the method provided here is rather of approximation. In spite of this, it may be useful in the practical approximation of sampling formula.

Hitherto we used sampling functions, which are of height unity at the main peak and have narrow breadth and low feet at both sides of the main peak, such as

$$\frac{g(z)}{(z-z_n)\cdot g'(z_n)} \qquad ,$$

with $g(z_n)=0$ (n=integer), and function g(z) having simple zeros at points $z=z_n$.
If one has interest, one can choose as g(z) in (4-1) the following functions:[29]

Now we shall propose to make an approximate sampling formula.[29] At first, we shall refer the following formula:

$$f(z)=\int_{-\infty}^{+\infty} f(\xi)\cdot\delta(z-\xi)\cdot d\xi \qquad , \qquad (7-1)$$

and approximate the right-hand side of (7-1), by replacing integration by summation and by taking $\Delta(z\text{-}z_n)$ instead of delta-function. Formally the expresion (7-1) becomes to be:

$$f(z)=\sum_n f(z_n)\cdot\Delta(z-z_n) \qquad , \qquad (7-2)$$

where sampling function $\Delta(z\text{-}z_n)$ is a continuous function of z, with main peak at points $z=z_n$ (n=integer), having low feet which extend to both sides of the main peak and vanishing at $|z| \rightarrow +\infty$. The sampling formula (7-2) corresponds to (4-1).

For example, we can take:

$$\Delta(z-z_n)\approx\exp\left[-\frac{(z-z_n)^{2m}}{2\sigma^2}\right] \qquad , \text{(m=positive integer, and } \sigma>0) \qquad (7-3)$$

$$\Delta(z-z_n)\approx\operatorname{sech}^m(z-z_n) \qquad , \text{(m=positive integer)} \qquad (7-4)$$

$$\Delta(z-z_n)\approx\text{Soliton-like functions, with } \Delta(0)=1 \text{ for } z=z_n \qquad (7-5)$$

$$\Delta(z-z_n)\approx 1/[\alpha \tan^2\{\beta(z-z_n)\}+1],(\alpha,\beta=\text{const}) \qquad (7-6)$$

and

$\Delta(z-z_n) \approx$ unit function with small breadth, etc. (7-7)

The approximate sampling formula (7-2) gives new formulae, by making use of (7-3) ~(7-7) etc.

§ 8. Numerical Examples

In practical applications, we are to take a truncated sampling fermulae (4-1)~(4-3), *i. e.* we are to make sampling at a **finite** number of sampling points. Here in § 8 we shall show some numerical examples of such truncated sampling formulae (4-1)～(4-3) as well as truncated formula (7-2).

In Figs. 2～10, a function f(z)=sin (2z) is sampled by means of g(z) used in (4-1)～(4-3). m=0,1, or 2, indicates that the zeros of g(z) are of 1st, 2nd, or 3rd order, respectdvely. Φ(z) is a calculated function making use of truncated formulae (4-1)～(4-3), being sampled at 6 points, which are marked with with small black circles "●". We can see that the truncated sampling formulae (4-1)～(4-3) are very useful for interpolation formulae. While they are not so convenient for extrapolation formulae in the region at the outside of the sampling interval.

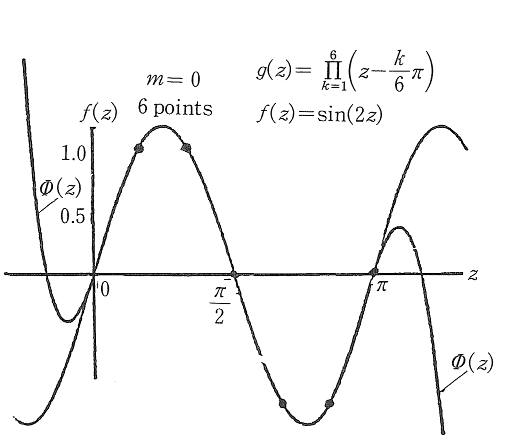

Fig.2.

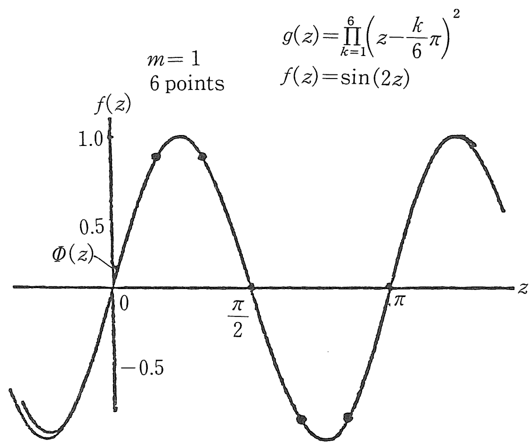

Fig.3.

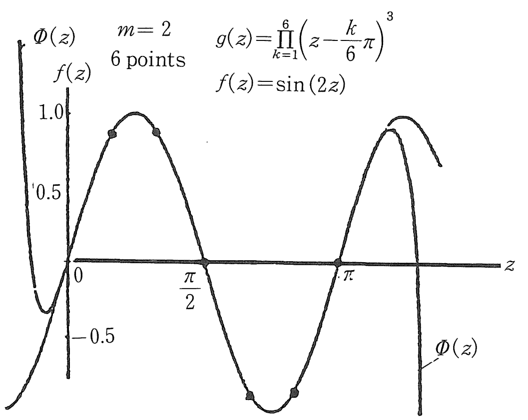

Fig.4.

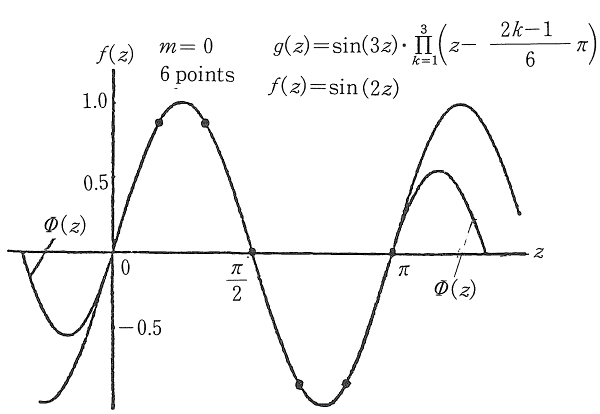

Fig.5.

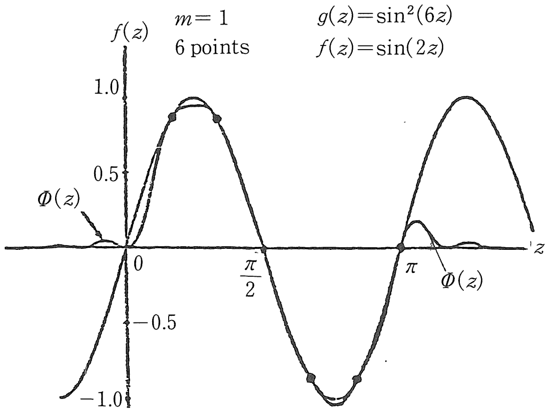

Fig.6.

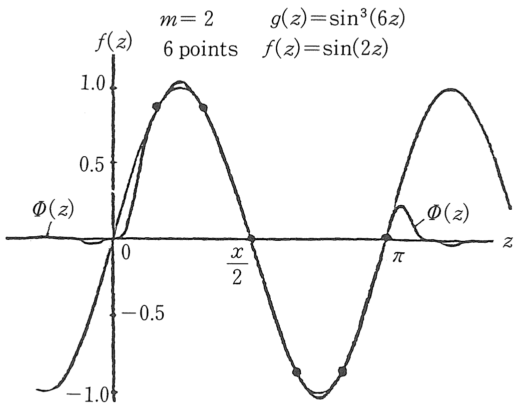

Fig.7.

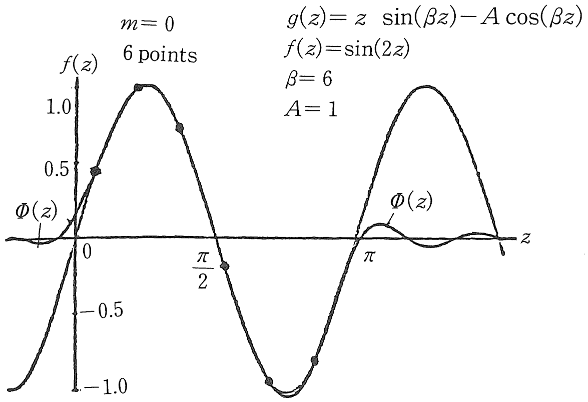

Fig.8.

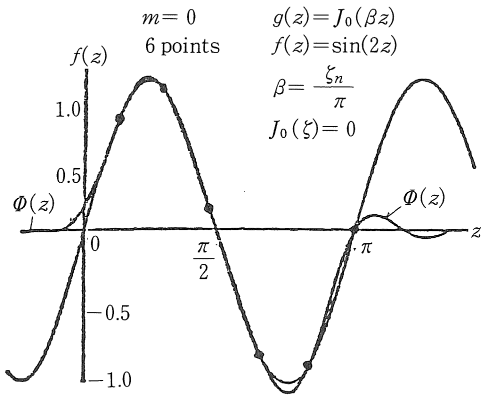

Fig.9.

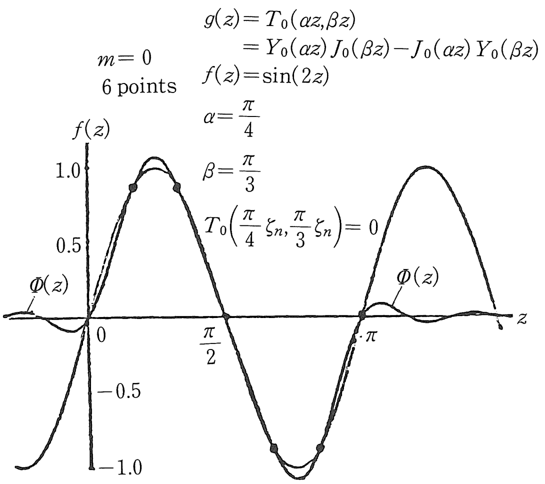

Fig.10.

In Figs. 11 and 12, the unit function f(z)=1 is sampled by means of (4-1) with $g(z)=\sin(6z)$ and $g(z)=\sin(3z)\cdot\prod_{k=1}^{3}(z-\frac{2k-1}{6}\pi)$, respectively. The latter shows fairely good approximation comparing with the former.

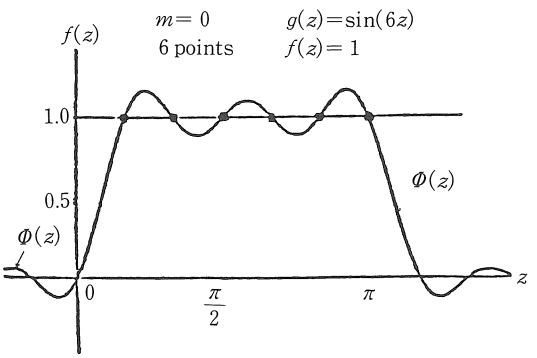

Fig.11.

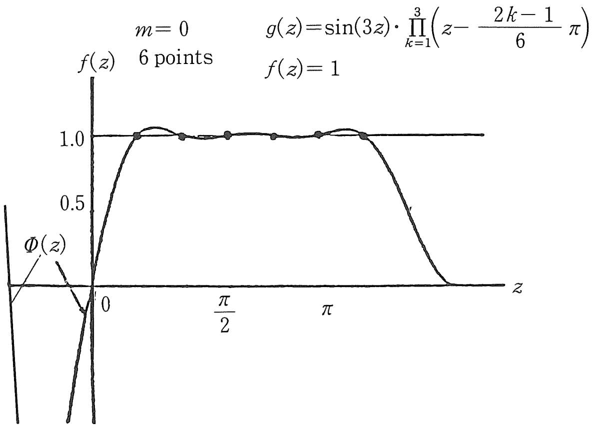

Fig.12.

For the sake of comparison of (7-2) with (4-1)~(4-3), we shall show numerical examples of the truncated sampling formula (7-2) in Figs.13~18. f(z)=sin (2z) shows the function to be sampled, and Φ (z) is the calculated function by truncated formula (7-2) with approximate sampling function $\Delta(z-z_n)$. Sampling points are marked with small circles "○". The values of parameters α and β taken here are also shown in each figure.

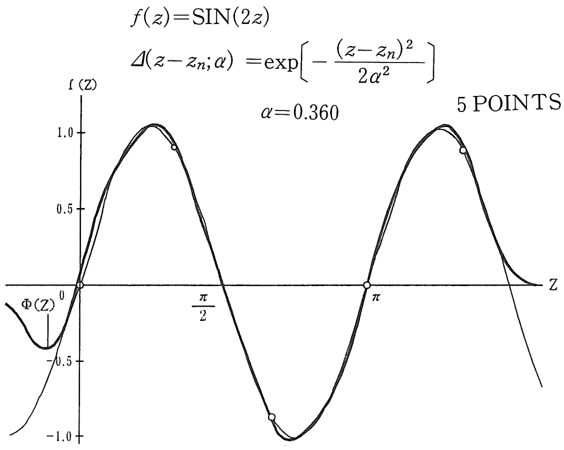

Fig.13.

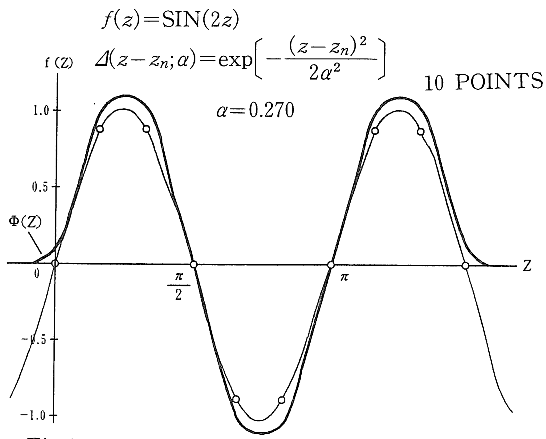

Fig.14.

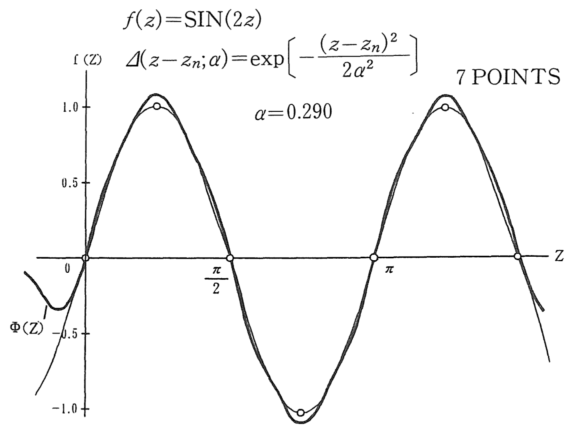

Fig.15.

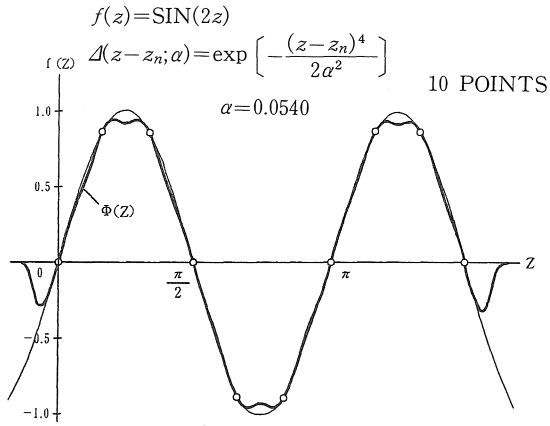

Fig.16.

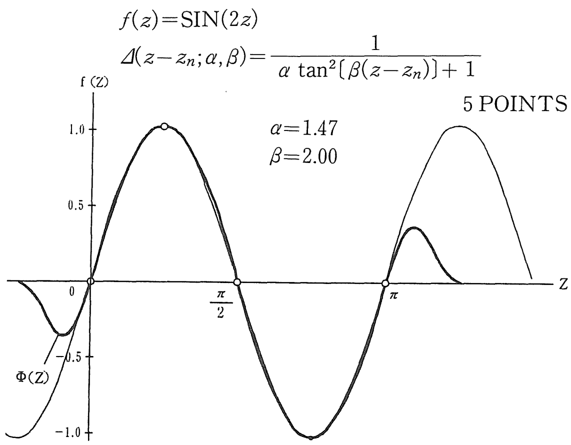

Fig.17.

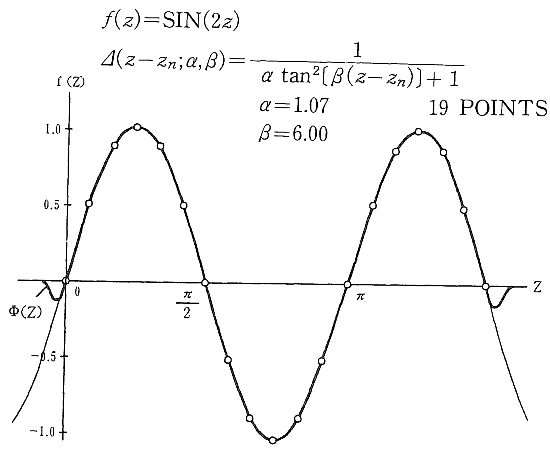

Fig.18.

While. in Figs. 19 and 20, the unit function f(z)=1 is sampled by truncated sampling formula (7-2), where Δ $(z-z_n)$ in (7-2)corresponds to $g(z)/[(z-z_n)\ g'(z_n)]$ in (4-1).

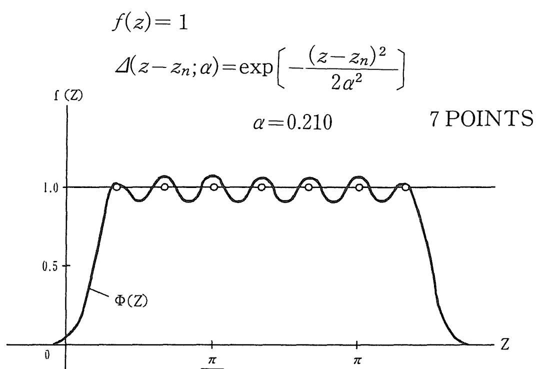

Fig.19.

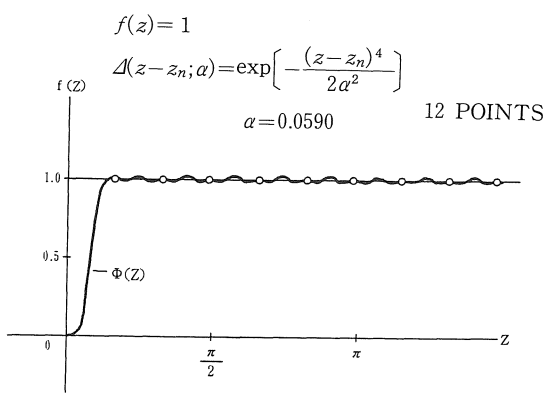

Fig.20.

In these Figs. 13～20, we see that the sampling function $\Delta(z-z_n)$ can be applicable to approximate the original sampled function f(z). The degree of precision in the approximation depends not only on the choice of the individual form of the function $\Delta(z-z_n)$ but also on the choice of sampling positions and values of parameters α and β.

§ 9. Concluding Remarks

In this paper, the authors presented the generalized sampling theorem (Theorem I), where an entire function f(z) can be expressed by means of sampled values f_s and sampled higher order derivatives $f_s^{(n)}$.

Many formulae hitherto obtained can be derived from the generalized sampling theorem presented here.

Some examples of the sampling functions are also proposed, which may be useful to construct an approximate interpolation formula.

In concluding this paper, the authors would like to emphasized that the generalized sampling theorem presented here may find good application in many fields of physics and engineering.

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