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A Note on p-Basis of a Regular Local Ring of Characteristic p

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標数pの正則局所環のp基底についての一注意

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Let $(R, \mathfrak{m})$ be a noetherian regular local ring with a quasi-coefficient field k of characteristic p and S be a subring of R containing R^p such that R is finite over S. The purpose of this paper is to prove that if $D_k(R)=D_S(R)$, than R have a p-basis over S.

1. Preliminaries.

In this paper, all rings are assumed to be commutative noetherian and to contain an identity element. Let p be always a prime number. Let R be a ring of characteristic p and R^p denote the subring $\{x^p | x \in R\}$. Let S be a subring of R. A subset B of R is said to be p-independent over S if the monomials $b_1^{e_1} \cdots b_n^{e_n}$, where $b_1, \cdots, b_n$ are distinct elements of B and $0 \leq e_i \leq p-1$, are linearly independent over $R^p[S]$. B is called a p-basis of R over S if it is p-independent over S and $R^p[S,B]=R$. Let R be a ring and let $\mathfrak{m}$ be an ideal of R. A ring R is called an $\mathfrak{m}$-adic ring if R is topologized by taking $\mathfrak{m}^n (n=1, 2, \cdots)$ as a fundamental system of neighborhoods of zero.

Let R be an $\mathfrak{m}$-adic ring. An R-module E is an $\mathfrak{m}$-adic R-module if E is endowed with the topology in which $\mathfrak{m}^n E (n=1, 2, \cdots)$ form a fundamental system of neighborhoods. An R-module E is said to be separated if $\bigcap_{n=1}^{\infty} \mathfrak{m}^n E=0$.

2. Lemmas.

We shall begin with a definition and then list the needed lemmas about $\mathfrak{m}$-adic differential module. The proofs of those lemmas are done by the standard arguments and we shall omit them.

Let R be a P-algebra and let $\mathfrak{m}$ be an ideal of R. We shall assume that R is an $\mathfrak{m}$-adic ring. We define the $\mathfrak{m}$-adic P-differential module of R, denoted by $\hat{D}_P(R)$, as the R-module satisfying the following conditions.

(1) There exists a P-derivation $\hat{d}_{R/P}$ from R into $\hat{D}_P(R)$.

(2) $\hat{D}_P(R)$ is generated over R by $\hat{d}_{R/P}x$, $x \in R$.

(3) $\hat{D}_P(R)$ is a separated $\mathfrak{m}$-adic R-module.

(4) For any P-derivation D of R into a separated $\mathfrak{m}$-adic R-module E, there exists an R-linear map h from $\hat{D}_P(R)$ into E such that

$$Dx=h(\hat{d}_{R/P}x) \qquad \text{for all } x \in R.$$

Lemma 1 ([1]Proposition 1). *Let R be a P-algebra and let $\mathfrak{m}$ be an ideal of R and assume that R is an $\mathfrak{m}$-adic ring. Let $D_P(R)$ be P-differential module of R. Then the $\mathfrak{m}$-adic P-differential module $\hat{D}_P(R)$ exists and is determined uniquely up to R-isomorphism. Moreover $\hat{D}_P(R)$ is given by*

$$\hat{D}_P(R)=D_P(R)/\bigcap_{n=1}^{\infty} \mathfrak{m}^n D_P(R).$$

Lemma 2 ([1]Corollary 2). *If $D_P(R)$ is a separated $\mathfrak{m}$-adic R-module, we have*

$$\hat{D}_P(R)=D_P(R)$$

Let R be an $\mathfrak{m}$-adic ring and let T be an $\mathfrak{n}$-adic R-algebra with a ring homomorphism $f: R \to T$, such that $f(1)=1$. We shall assume that f satisfies the condition

$$f(\mathfrak{m}) \subset \mathfrak{n}. \tag{1}$$

Lemma 3 ([1]Theorem 3). *Let R be an $\mathfrak{m}$-adic ring and let R^* be the $\mathfrak{m}$-adic completion of R. Let T be an R-algebra satisfying the condition* (1).
Assume that $(T, \mathfrak{n})$ is a Zariski ring and $\hat{D}_R(T)$ is a finite T-module and let T^* be the $\mathfrak{n}$-adic completion of T. Then we have

$$\hat{D}_{R^*}(T^*) = \hat{D}_R(T^*) = T^* \otimes_T \hat{D}_R(T).$$

Lemma 4 ([3]§38 Proposition). *Let R be a local ring of characterisitic p and S be a subring of R containing R^p such that R is finite over S. If $D_S(R)$ is a free R-module with $dx_1, \cdots, dx_r$ ($x_i \in R$) as a basis, then $x_1, \cdots, x_r$ form a p-basis of R over S.*

Lemma 5 ([1]Proposition 10). *Let R be a formal power series ring in n-variables $X_1, \cdots, X_n$ over a ring S and let $\mathfrak{m}$ be the ideal of R generated by $(X_1, \cdots, X_n)$. Then the $\mathfrak{m}$-adic S-differential module $\hat{D}_S(R)$ is free module of rank n.*

3. Results.

Proposition 6. *Let $(R, \mathfrak{m})$ be a regular local ring with a quasi-coefficient field k and S be a subring of R. Assume that we have $D_k(R) = D_S(R)$. If $D_S(R)$ is a finite R-module, then $D_S(R)$ is a free R-module.*

Proof. Let R^* denote the $\mathfrak{m}$-adic completion of R. Since R is a regular local ring , R^* is regular. Let us put $\dim R = r$. Since k also is a quasi-coefficient field of R^* and $(R^*)^* = R^*$, R^* contains a coefficient field K containing k. Therefore, R^* is expressed as a formal power series ring $K[\![X_1, \cdots, X_r]\!]$, where $K = R^*/\mathfrak{m}^* = R/\mathfrak{m}$. Since $D_S(R)$ is finite, $D_S(R)$ is separated and we have $\hat{D}_S(R) = D_S(R)$ by Lemma 2. Then, by Lemma 3, we have

$$\hat{D}_S(R^*) \cong R^* \otimes_R \hat{D}_S(R) \cong R^* \otimes_R \hat{D}_S(R) \tag{2}$$

From our assumption $D_k(R) = D_S(R)$ and (2), we have $\hat{D}_S(R^*) \cong \hat{D}_k(R^*)$. Since K is formally etale over k, we have $D_k(K) = 0$. Therefore, we see that $D_k(R^*)$ and $D_K(R^*)$ are isomorphic by Theorem 57 of [3] and we have $\hat{D}_k(R^*) \cong \hat{D}_K(R^*)$. So, by Lemma 5, $\hat{D}_S(R^*) \cong \hat{D}_K(R^*) = \hat{D}_K(K[\![X_1, \cdots, X_r]\!])$ is a free module of rank r and since R is faithfully flat, $D_S(R) = D_k(R)$ is free. Thus the proof is complete,

Theorem 7. *Let $(R, \mathfrak{m})$ be a local ring with coefficient field k of characteristic p and S be a subring of R containing R^p such that R is finite over S. Assume that we have $D_k(R) = D_S(R)$. If R is regular, then R has a p-basis over S.*

Proof. Since R is finite over S, $D_S(R)$ is a finite R-module. Therefore, our theorem is proved by Proposition 6 and Lemma 4.

References

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