Examples of Some Homeomorphisms on Real 2-Sphere

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実2次元球面上のある同相写像の例

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We consider some class of measure-preserving homeomorphisms on real 2-sphere. A classification of these homeomorphisms has been given already. In this paper, along this line, we give examples of such homeomorphisms of order ∞ by the method of construction available for some general cases.

§ 1. Introduction.

Let S^2 be a real 2-sphere. We consider the homeomorphisms on S^2 satisfying the following Conditions (C).

(1) The homeomorphism f on S^2 has the fixed points N (the north pole) and S (the south pole), or substitutes these two points each other. Further, f is of class C^1 in S^2 except for N and S.

(2) f is measure-preserving, that is, f preserves the uniform probability measure on S^2.

Among these hemomorphisms satisfying Conditions (C), we denote these preserving each latitude by L, and others by NL. Concerning L, a classification of the discrete topological dynamical systems generated by $f \in L$ is given in Nagaya 〔2〕, by considering the cyclic groups obtained by the systems. And concerning NL, a similar classification is given also in 〔2〕, and the existece of $f \in NL$ of order ∞ is confirmed in Hashimoto 〔1〕 by constructing an example. The purpose of this paper is to give further examples of $f \in NL$ of order ∞ by using the ideas in 〔1〕.

§ 2. Construction of examples.

Now, we shall give the following

THEOREM. There exist some homeomorphisms in NL of order ∞.

PROOF. Let S^2 be as in § 1. We may assume S^2 the unit sphere in the Euclidean xyz-space with its center at the origin O. We shall introduce into S^2 the polar coordinate systems. That is, for P $\in S^2$, denoting by θ the angle between the z-axis and OP and by ϕ the angle between x-axis and OP′, where P′ is a projection of P onto xy-plane, we represent P $\in S^2$ by (θ,ϕ). Then, denoting the coordinate systems in the image sphere by (Θ,Φ), we can represent a homeomorphism on S^2 by the mapping $(\theta,\phi) \rightarrow (\Theta,\Phi)$ with appropriate conditions.

Now, the area element of S^2 is given by $\sin\theta \, d\theta \wedge d\phi$ and that of the image sphere is given by $\sin\Theta \, d\Theta \wedge d\Phi$. So that, if the mapping $(\theta,\phi) \rightarrow (\Theta,\Phi)$ with appropriate conditions satisfies

$$\left(\frac{\partial\Theta}{\partial\theta}\frac{\partial\Phi}{\partial\phi}-\frac{\partial\Theta}{\partial\phi}\frac{\partial\Phi}{\partial\theta}\right)\sin\Theta = \pm\sin\theta,$$

it is measure-preserving. Here, setting $\cos\theta = \lambda$ and $\cos\Theta = \Lambda$, the condition above reduces to

$$\frac{\partial\Lambda}{\partial\lambda}\frac{\partial\Phi}{\partial\phi}-\frac{\partial\Lambda}{\partial\phi}\frac{\partial\Phi}{\partial\lambda} = \pm 1.$$

Therefore, we may only construct the mapping $(\lambda,\phi) \rightarrow (\Lambda,\Phi)$ satisfying the following Conditions (C′).

(1′) $(\Lambda(\lambda,\phi), \Phi(\lambda,\phi))$ is a continuous mapping of $\{(\lambda,\phi) \mid -1 \leqq \lambda \leqq 1, 0 \leqq \phi \leqq 2\pi\}$ into $\{(\Lambda,\Phi) \mid -1 \leqq \Lambda \leqq 1\}$ and a one-to-one mapping of $\{(\lambda,\phi) \mid -1 < \lambda < 1, 0 \leqq \phi \leqq 2\pi\}$ into $\{(\Lambda,\Phi) \mid -1 < \Lambda < 1\}$, which satisfies the boundary conditions $\Lambda(1,\phi) = 1$, $\Lambda(-1,\phi) = -1$. $\Lambda(\lambda,2\pi) = \Lambda(\lambda,0)$ and $\Phi(\lambda,2\pi) = \Phi(\lambda,0) + 2\pi$. Further, it is of class C^1 in $\{(\lambda,\phi) \mid -1 < \lambda < 1, 0 \leqq \phi \leqq 2\pi\}$.

(2′) $\dfrac{\partial\Lambda}{\partial\lambda}\dfrac{\partial\Phi}{\partial\phi} - \dfrac{\partial\Lambda}{\partial\phi}\dfrac{\partial\Phi}{\partial\lambda} = 1.$

Here, we remark that this is the case where the homeomorphism has the fixed points N and S and is orientation preserving.

To construct the mapping, we first confine the image region to $\{(\Lambda,\Phi) \mid -1 \leqq \Lambda \leqq 1, 0 \leqq \Phi \leqq 2\pi\}$. Let $\varkappa(\phi)$ be a real-valued function in $-\infty < \phi < \infty$ of class C^1 with period 2π satisfying $\varkappa(0) = \varkappa(2\pi) = 0$, $|\varkappa'(\phi)| \leqq 1$ and $\varkappa'(0) \neq 0$.
And we set the mapping as

$$(\Lambda,\Phi) = (\Lambda(\lambda,\phi), -\varkappa(\phi)\Lambda(\lambda,\phi) + \phi),$$

that is, we set the mapping so that the line segments in the $\lambda\phi$-plane connecting $(-1,\phi)$ and $(1,\phi)$ are set-wise mapped onto the line segments in the $\Lambda\Phi$-plane connecting $(-1,\phi+\varkappa(\phi))$ and $(1,\phi-\varkappa(\phi))$. Here, according to the conditions $\varkappa(0) = \varkappa(2\pi) = 0$ and $|\varkappa'(\phi)| \leqq 1$, $\phi + \varkappa(\phi)$ and $\phi - \varkappa(\phi)$ are increasing functions of $[0, 2\pi]$ onto $[0, 2\pi]$. Now, to determine $\Lambda(\lambda,\phi)$, we use the above condition (2′), which implies

$$\{-\varkappa'(\phi)\Lambda + 1\}\frac{\partial\Lambda}{\partial\lambda} = 1.$$

For solving this differential equation, we assume $\varkappa'(\phi) \neq 0$ and set $-\kappa'(\phi)\Lambda + 1 = \tilde{\Lambda}$. Then, we have

$$\tilde{\Lambda}\frac{\partial\tilde{\Lambda}}{\partial\lambda} = -\varkappa'(\phi).$$

Integrating this equation for λ, we have

$$\frac{\tilde{\Lambda}^2}{2} = -\varkappa'(\phi)\lambda + \frac{\gamma(\phi)}{2},$$

where $\gamma(\phi)$ is a function of ϕ determined later. Now, as $|\varkappa'(\phi)| \leqq 1$ and $-1 \leqq \Lambda \leqq 1$, we have $\tilde{\Lambda} \geqq 0$, so that,

$$\tilde{\Lambda} = \sqrt{-2\varkappa'(\phi)\lambda + \gamma(\phi)}.$$

Here, using the boundary condition $\Lambda(1,\phi) = 1$, $\Lambda(-1,\phi) = -1$ in (1′), we have

$$\gamma(\phi) = 1 + \{\varkappa'(\phi)\}^2,$$

and

$$\Lambda = \frac{\sqrt{1 - 2\varkappa'(\phi)\lambda + \{\varkappa'(\phi)\}^2} - 1}{-\varkappa'(\phi)}.$$

Therefore, we have

$$\Lambda = \frac{2\lambda - \varkappa'(\phi)}{1 + \sqrt{1 - 2\varkappa'(\phi)\lambda + \{\varkappa'(\phi)\}^2}},$$

which is also valid in the case $\varkappa'(\phi) = 0$, and

$$\Phi = \frac{-2\varkappa(\phi)\lambda + \varkappa(\phi)\varkappa'(\phi)}{1 + \sqrt{1 - 2\varkappa'(\phi)\lambda + \{\varkappa'(\phi)\}^2}} + \phi.$$

We can easily see that this mapping (Λ, Φ) satisfies other conditions in (1′).

The rest of the proof is to show that the above (Λ,Φ) is in NL and of order ∞. Considering the image of $(\lambda,\phi) = (0,\phi)$ $(0 \leqq \phi \leqq 2\pi)$,

$$(\Lambda,\Phi) = \left(\frac{-\varkappa'(\phi)}{1 + \sqrt{1 + \{\varkappa'(\phi)\}^2}}, -\varkappa(\phi)\Lambda + \phi\right),$$

we can see that the homeomorphism represented by this mapping does not preserve the latitude.

Further, considering the image of $(\lambda,\phi) = (\lambda, 0)\ (-1 \leqq \lambda \leqq 1)$,

$$(\Lambda,\Phi) = \left(\frac{\sqrt{1-2\varkappa'(0)\lambda+\{\varkappa'0\}^2}-1}{-\varkappa'(0)}, 0\right),$$

we can see that it is of order ∞. Q.E.D.

We can give examples of $\varkappa(\phi)$ such as $\sin\phi$, whose case is treated in [1], $a\sin\phi$ $(0 < a \leqq 1)$, $\frac{1}{n}\sin(n\phi)$, $\frac{1}{2}\{\sin\phi+\frac{1}{2}\sin(2\phi)\}$ and so on.

REFERENCES

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