An Example of Some Homeomorphism on Real 2-Sphere

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実２次元球面上のある同相写像の１例

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Measure-preserving homeomorphisms on a topological measure space have some interesting and important properties. In this paper, we show an example of some non-latitude-preserving homeomorphism on real 2-sphere.

§ 1. Introduction.

Let S^2 be a real 2-sphere. We consider the homeomorphisms on S^2 satisfying the following conditions.

Conditions (C).

(1) The homeomorphism f on S^2 has the fixed points N (the north pole) and S (the south pole), or substitutes N for S and S for N. Further, f is of class C^1 in S^2 except for N and S.

(2) f preserves the uniform probability measure on S^2.

Among these homeomorphisms satisfing the Conditions (C), we denote those preserving each latitude by L, and others by NL. Concerning L, a classification of the discrete topological dynamical systems generated by $f \in L$ has been given by H. Nagaya 〔2〕, noticing the cyclic groups obtained by the systems.

In this paper, we shall give the following theorem concerning NL.

Theorem.

There exists at least one f in NL of order ∞.

We shall give proof of the theorem by constructing an example of such a homeomorphism. We also remark that a family of such homeomrphisms can be obtained in Y. Hashimoto and H. Nagaya 〔1〕.

§ 2. Proof of Therem.

We can assume S^2 the unit sphere in the Euclidean xyz-space with the center at the origin O. We shall introduce into S^2 the polar coordinate systems. That is, for $P \in S^2$, we denote by θ the angle between the z-axis and OP, and by φ the angle between the x-axis and OP′, where P′ is a projection of P on xy-plane. Further, we denote the same quantities in the image sphere by Θ and Φ respectively.

In these circumstances, setting $\cos\theta=\lambda$ and $\cos\Theta=\Lambda$, we can represent the homeomorphism satisfying the Conditions (C) as the mapping $(\lambda, \varphi) \to (\Lambda, \Phi)$ which satisfies the following conditions.

(1) $(\Lambda(\lambda,\varphi),\Phi(\lambda,\varphi))$ is a one-to-one continuous mapping of $-1\leqq\lambda\leqq1$, $0\leqq\varphi\leqq2\pi$ into $-1\leqq\Lambda\leqq1$ satisfying $\Lambda(1,\varphi)=1$, $\Lambda(-1,\varphi)=-1$, $\Lambda(\lambda,2\pi)=\Lambda(\lambda,0)$ and $\Phi(\lambda,2\pi)=\Phi(\lambda,0)+2\pi$. Further, $(\Lambda(\lambda,\varphi),\Phi(\lambda,\varphi))$ is of class C^1 in $-1<\lambda<1$, $0\leqq\varphi\leqq2\pi$.

(2) $\dfrac{\partial\Lambda}{\partial\lambda}\dfrac{\partial\Phi}{\partial\varphi}-\dfrac{\partial\Lambda}{\partial\varphi}\dfrac{\partial\Phi}{\partial\lambda}=1$ in $-1<\lambda<1$, $0\leqq\varphi\leqq2\pi$.

Here, we consider the case where the homeomorphism has the fixed points N and S and is orientation-preserving.

Now, we shall construct the mapping satisfying the above conditions. First we confine the image region to $-1\leqq\Lambda\leqq1$, $0\leqq\Phi\leqq2\pi$, and then set the mapping.

$$(\Lambda,\Phi)=(\Lambda(\lambda,\varphi),-\Lambda(\lambda,\varphi)\sin\varphi+\varphi).$$

That is, we set the mapping so that the line segments in the $\lambda\varphi$-plane connecting $(-1,\varphi)$ and $(1,\varphi)$ are set-wise mapped onto the line segments in the $\Lambda\Phi$-plane connecting $(-1,\varphi+\sin\varphi)$ and $(1,\varphi-\sin\varphi)$. Then, the above condition in (2) implies

$$(-\Lambda\cos\varphi+1)\frac{\partial\Lambda}{\partial\lambda}=1,$$

and for solving this differential equation, we set $-\Lambda\cos\varphi+1=\tilde{\Lambda}$ in case $\cos\varphi\neq0$, hence

$$\tilde{\Lambda}\frac{\partial\tilde{\Lambda}}{\partial\lambda}=-\cos\varphi.$$

By integrating this equation for λ and noticing $\tilde{\Lambda}\geqq0$, we have

$$\tilde{\Lambda}=\sqrt{-2\lambda\cos\varphi+C(\varphi)},$$

therefore,

$$\Lambda=\frac{1-\sqrt{-2\lambda\cos\varphi+C(\varphi)}}{\cos\varphi},$$

where $C(\varphi)$ is an arbitrary function of class C^1. Here, accrding to the boundary conditions in (1), we have,

$$C(\varphi)=1+\cos^2\varphi,$$

and then,

$$\Lambda=\frac{2\lambda-\cos\varphi}{1+\sqrt{1-2\lambda\cos\varphi+\cos^2\varphi}},$$

which is also valid in case $\cos\varphi=0$. Thus, we obtain the mapping

$$(\Lambda,\Phi)=\Big(\frac{2\lambda-\cos\varphi}{1+\sqrt{1-2\lambda\cos\varphi+\cos^2\varphi}},\ \frac{-2\lambda\sin\varphi+\sin\varphi\cos\varphi}{1+\sqrt{1-2\lambda\cos\varphi+\cos^2\varphi}}+\varphi\Big).$$

Now, considering the image of $\lambda=0$, $0\leqq\varphi\leqq 2\pi$

$$(\Lambda,\Phi)=\Big(\frac{-\cos\varphi}{1+\sqrt{1+\cos^2\varphi}},\ \frac{\sin\varphi\cos\varphi}{1+\sqrt{1+\cos^2\varphi}}+\varphi\Big),$$

we can see that the homeomrphism represented by this mapping does not preserve the latitude. Also, considering the image of $-1\leqq\lambda\leqq 1$, $\varphi=0$

$$(\Lambda,\Phi)=(1-\sqrt{2-2\lambda},\ 0),$$

we can see that it is of order ∞. Q.E.D.

REFERENCES

〔1〕 Y. Hashimoto and H. Nagaya, Examples of some homeomorphisms on real 2-sphere, in preparation.

〔2〕 H. Nagaya, On a classification of some discrete topological dynamical systems on real 2-sphere, in preparation.

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