Identity Elements in Rings

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As is well-known, (one-sided or two-sided) identity elements in rings play an important role in the thory of rings and modules. The purpose of this paper is to consider several conditions for a ring to have identity elements.

§ 1

Definitions. Throughout R will represent an associative ring. An element $e \in$ R is called a right (left) identity if $xe = x$ ($ex = x$) holds for any $x \in$ R. If e is both a right identity and left identity, e is called an identity and denoted by 1. When R is a ring with 1, a right R-module M is called unitary if $m1 = m$ holds for any $m \in$ M.

When S is a subset of R, A_l (S) denotes the left annihilator $\{x \in R \mid xS = 0\}$. Similarly A_r (S) is the right annihilator.

Let A be a ring with 1 and N be a unitary right A-module. The Abelian group $A \oplus N$ with the multiplication

$$(a_1,n_1)(a_2,n_2) = (a_1a_2,n_1a_2)$$

is a ring, which is denoted by $[A; N_A]$. Naturally N is regarded as an ideal of $[A; N_A]$ by the monomorphism $n \longmapsto (0, n)$. Also A is regarded as a right ideal of $[A; N_A]$ by $a \longmapsto (a, 0)$.

Lemma 1.1 (1) $(1, n)$ is a right identity of $[A; N_A]$ for any $n \in N$.

(2) $N = A_r([A;N_A])$.

(3) A is isomorphic to the left $[A; N_A]$-endomorphism ring of $[A; N_A]$.

Proof. As (1) and (2) are easy, we shall show only (3). Let f be a left $[A; N_A]$-endomorphism of $[A; N_A]$, then one will easily see that $f((1, 0)) = (a, 0)$ for some $a \in A$. Let ϕ be the mapping $f \longmapsto a$. As is easily verified, ϕ is a ring homomorphism.

Conversely, for any $a \in A$, let f be the endomorphism of $[A; N_A]$ defined by $f((x, n)) = (xa, na)$. Denote the mapping $a \longmapsto f$ by ψ, then $\phi \circ \psi = \psi \circ \phi = id$. This completes the proof.

Theorem 1.2 If R has a right identity, then there exist a ring A with identity and a unitary right A-module N such that $R \cong [A; N_A]$. A and N_A are uniquely determined up to isomorphism.

Proof. Let e be a right identity of R. Then $R = eR \oplus A_r(e)$ as right R-modules. If we put $A = eR$, A is a ring with e an identity and $A_r(e) = A_r(R) = N$ is naturally regarded as a right A-module. Any $r \in R$ is uniquely written as $r = a + n$ ($a \in A$, $n \in N$). The mapping $\varphi: r \longmapsto (a, n)$ gives an isomorphism from R to $[A; N_A]$. The uniqueness of A and N_A is clear from Lemma 1.1.

Corollary 1.3 If R has a right identity and $A_r(R) = 0$, then R has an identity.

Corollary 1.4 If R has a unique right identity, then it is an identity.

For, both of these conditions imply $N = 0$.

Since $A_r(R)$ is contained in the Jacobson radical of R, if a semisimple ring has a right identity, then it is an identity.

Theorem 1.5 (cf. [1] § 6) If $[A; N_A]$ is left Artinian, then A is left Artinian and N consists of only finitely many elements.

Proof. For any left ideal L of A, $[L; N] = \{(a, n) \in [A; N_A] \mid a \in L\}$ is a left ideal of $[A; N_A]$. From this we can see that A is left Artinan.

For any Abelian subgroup N′ of N, $[0; N'] = \{(0, n) \in [A; N_A] \mid n \in N'\}$ is a left ideal of $[A; N_A]$. It follows that Abelian subgroups of N satisfy the descending chain condition.

Let x be an arbitrary element of N. If we suppose that the additive order of x is infinite, we get a strictly descending chain of Abelian subgroups of N

$$Zx \supsetneqq 2Zx \supsetneqq 2^2Zx \supsetneqq \cdots\cdots$$

This is a contradiction, so any element of N has a finite order. It follows that

$$N = N_{p_1} \oplus N_{p_2} \oplus \ldots\ldots \oplus N_{p_t},$$

where each N_{p_i} is a primary Abelian subgroup belonging to a prime p_i and $p_1, p_2, \ldots, p_t$ are distinct primes. Without any loss of generality, we can suppose $N = N_{p_i}$, that is, there exists a prime $p = p_1$ such that the order of any element of N is a power of p.

Let us put $N^{(j)} = \{x \in N \mid p^j x = 0\}$ for each positive integer j, then

$$N^{(1)} \subseteqq N^{(2)} \subseteqq \ldots\ldots \subseteqq N^{(m)} \subseteqq \ldots\ldots$$

is an ascending chain of Abelian subgroups of N and $N = \bigcup_{i=1}^{\infty} N^{(i)}$. Suppose there exists a strictly increasing sequence of positive integers $e_1 < e_2 < \ldots . < e_n < \ldots .$ such that $N^{(e_1)} \subsetneqq N^{(e_2)} \subsetneqq \ldots . \subsetneqq N^{(e_n)} \subsetneqq \ldots .$. Regarding that each $N^{(j)}$ is a right A-submodule of N, we get a strictly descending infinite chain of left ideals of A

$$p^{e_1} A \supsetneqq p^{e_2} A \supsetneqq \ldots . \supsetneqq p^{e_n} A \supsetneqq \ldots .$$

This contradicts that A is left Artinian. It follows that there exists a positsve integer k such that $N^{(k)} = N$.

$$0 = N^{(0)} \subseteqq N^{(1)} \subseteqq N^{(2)} \subseteqq \ldots \subseteqq N^{(k)} = N$$

is a chain of Abelian subgroups of N, where each $N^{(j)}/N^{(j-1)}$ $(1 \leqq j \leqq k)$ is a finite direct sum of cyclic groups of order p by the descending chain condition. Hence N is a finite set.

§ 2

Definitions. When R is a ring, J(R) denotes the Jacobson radical of R, which means the intersection of all modular, maximal left ideals of R (cf. [6] Chapter III). $R^{\times}$ will represent the multiplicative semigroup of R. Also,

$B(R) = \{a \in R \mid a \in Ra\}$, $B'(R) = \{a \in R \mid a \in aR\}$,
$S(R) = \{a \in R \mid R = Ra\}$, and $T(R) = \{a \in R \mid A_l(a) = 0\}$.

A left ideal L of R is called to be small if L + M is a proper left ideal whenever M is a proper left ideal of R.

Lemma 2.1 (1) B(R) is a (semigroup-theoretic) right ideal of $R^{\times}$.

(2) S(R) and T(R) are subsemigroups of $R^{\times}$.

(3) $S(R) \subseteqq B(R)$.

Theorem 2.2 R has a right identity if and only if $B(R) \cap T(R) \neq \phi$.

Proof. Let $B(R) \cap T(R) \neq \phi$ and $a \in B(R) \cap T(R)$. Then there exists $e \in R$ such that $a = ea$. Let x be an arbitrary element of R, then

$$(x - xe)a = x(a - ea) = 0.$$

It follows that $x = xe$, hence e is a right identity.

Since every element of J(R) is quasi-regular, we can easily see that J(R) is a small left ideal if R has a right identity. The converse is not true in general, but the following fact is known.

Thenrem 2.3 ([2] , Satz 2) R has a right identity if and only if the following three conditions are satisfied.

(1) R/J(R) has an identity.

(2) J(R) is a small left ideal.

(3) B′(R) = R.

In case R is left or right Noetherian, the following is known.

Theorem 2.4 ([8]) When R is left or right Noetherian, R has a right identity if and only if B′(R) = R.

We can give an another proof in case R is left Noetherian. Assume that R is left Noetherian and B′(R) = R. Let M be the set of all left ideals I of R which satisfies the following condition:

(∗) There exists some e (depending on I) $\in R$ such that $xe = x$ for any $x \in I$.

Since M is not empty, M has a maximal element I^*. There exists $e^* \in R$ which satisfies $xe^* = x$ for any $x \in I^*$. Let us assume that $I^* \neq R$, then there exists $a \in R$ with $a \notin I^*$. $K = I^* + Ra + Za$ is a left ideal of R which contains I^* properly. We can choose $e \in R$ such that $(ae^* - a)e = ae^* - a$. If we put $e' = e^* + e - e^* e$, then for any element $y = x + ra + za$ ($x \in I^*$, $r \in R$, $z \in Z$) of K, it holds that

$$\begin{aligned} ye' &= x(e^* + e - e^*e) + ra(e^* + e - e^*e) + \\ &\quad za(e^* + e - e^*e) \\ &= xe^* + xe - xe^*e + r(ae^* + ae - ae^*e) + \\ &\quad z(ae^* + ae - ae^*e) \\ &= y. \end{aligned}$$

It follows that $K \in M$. This contradicts the maximality of I^*. Consequently $I^* = R$, hence R has a right identity.

Definition. An element a of R will be called a right multiplicator if there exists a fixed integer n such that $xa = nx$ holds for any $x \in R$. M(R) will represent the set of all right multiplicators of R, which forms a subring of R.

Theorem 2.5 ([5] , Satz 3.1) R has a right identity if and only if the following two conditions are satisfied.

(1) For any homomorphic image R′ of R, it holds that $A_l(R') = 0$.

(2) $M(R) \cap T(R) \neq \phi$.

§ 3

We consider two conditions concerning an element $a \in R$.

(A) $Ra = R$ (i.e. $a \in S(R)$)

(B) $A_l(a) = 0$ (i.e. $a \in T(R)$)

These two conditions are independent in general.

Example 1. Let R be a commutative integral domain (for instance, Z). If a is different from 0, then (B) holds, though (A) may not.

Example 2. Let V be a vector space over a field k of countably infinite dimension with a basis $\{e_1, e_2, \ldots, e_n, \ldots\}$. Let R be the endomorphism ring of V. We define $a \epsilon R$ by $e_i \longmapsto e_{i+1}$ $(1 \leqq i < \infty)$. Also $b \epsilon R$ is defined by $e_1 \longmapsto 0$ and $e_i \longmapsto e_{i-1}$ $(2 \leqq i < \infty)$. Then clearly we obtain $ba = 1$ (identity map), hence $Ra = R$. If we define $c \epsilon R$ by $e_1 \longmapsto e_1$ and $e_i \longmapsto 0$ $(2 \leqq i < \infty)$, then $ca = 0$, so $A_1(a) \neq 0$.

But we shall show that (A) and (B)are equivalent if R is both left Noetherian and left Artinian.

Theorem 3.1 If $S(R) \neq \phi$, the following conditions are equivalent.

(1) $S(R) = T(R)$.

(2) A left R-endomorphism $f: R \longrightarrow R$ is injective when and only when it is surjective.

(3) (i) R is the only left ideal of R which is isomorphic to R as left R-modules,and (ii) $A = 0$ is the only left ideal which satisfies $R/A \cong R$ as left R-modules.

Proof. (1) $\longrightarrow$ (2) Choose $a \epsilon S(R)$, and let $f: R \longrightarrow R$ be an injective left R-endomorphism. If we put $f(a) = b$, then $A_l(b) = 0$, hence we get $Rb = R$. Let r be an arbitrary element of R, then there exists $s \epsilon R$ such that $r = sb$. So $r = sf(a) = f(sa)$, which implies that f is surjective.

Next suppose that $f: R \longrightarrow R$ is a surjective left R-endomorphism. Since $R = f(R) = f(Ra) = Rb$, $A_l(b) = 0$. Let x be an element of $Ker(f)$. There exists $y \epsilon R$ such that $x = ya$, so $0 = f(x) = f(ya) = yf(a) = yb$. It follows that $y = 0$, hence f is injective.

(2) $\longrightarrow$ (3) Let A be a left ideal of R and $\varphi: R \longrightarrow A$ be a left R-isomorphism. If we denote the natural injection from A to R by j, then $j \circ \varphi: R \longrightarrow R$ is injective, hence surjective. That is, $A = R$.

Next suppose that A is a left ideal of R and there exists a left R-isomorphism $\psi: R/A \longrightarrow R$. Let $\pi: R \longrightarrow R/A$ be the natural projection, then $\psi \circ \pi: R \longrightarrow R$ is surjective. Hence it is injective and $A = Ker(\psi \circ \pi) = 0$.

(3) $\longrightarrow$ (1) is clear from $Ra \cong R/A_l(a)$

Lemma 3.2(1) If a left R-module M satisfies the descending chain condition, then any injective left R-endomorphism of M is surjective.

(2) If a left R-module M satisfies the ascending chain condition, then any surjective left R-endomorphism of M is injective.

Proof. (1) Let $\varphi: M \longrightarrow M$ be an injective endomorphism. Since

$$M = \varphi^0(M) \supseteqq \varphi(M) \supseteqq \varphi^2(M) \supseteqq \ldots,$$

by the descending chain condition there exists $n \geqq 0$ such that $\varphi^n(M) = \varphi^{n+1}(M)$: suppose n is the least such integer. Let us assume $n \geqq 1$. If $m \epsilon \varphi^{n-1}(M)$, there exists $m' \epsilon M$ such that $m = \varphi^{n-1}(m')$. Also there exists $m'' \epsilon M$ such that $\varphi(m) = \varphi^n(m') = \varphi^{n+1}(m'')$. Then $\varphi(m - \varphi^n(m'')) = 0$, which follows that $m = \varphi^n(m'')$, since φ is injective. So $\varphi^{n-1}(M) = \varphi^n(M)$, which contradicts the definition of n. Therefore, $M = \varphi(M)$.

(2) Let $\psi: M \longrightarrow M$ be a surjective endomorphism. Since

$$0 = Ker(\psi^0) \subseteqq Ker(\psi) \subseteqq Ker(\psi^2) \subseteqq \ldots,$$

there exists $n \geqq 0$ such that $Ker(\psi^n) = Ker(\psi^{n+1})$: suppose n is the least such integer.Let us assume $n \geqq 1$. If $a \epsilon Ker(\psi^n)$, there exists $b \epsilon M$ such that $a = \psi(b)$. Since $\psi^n(a) = \psi^{n+1}(b) = 0$, $b \epsilon Ker(\psi^{n+1}) = Ker(\psi^n)$. Then $0 = \psi^n(b) = \psi^{n-1}(\psi(b)) = \psi^{n-1}(a)$, which means $a \epsilon Ker(\psi^{n-1})$. So $Ker(\psi^{n-1}) = Ker(\psi^n)$, a contradiction. Therefore $Ker(\psi) = 0$.

From this, we can get the following:

Theorem 3.3 If R is both left Noetherian and left Artinian, then $S(R) = T(R)$.

Proof. For each $a \epsilon R$, we only have to apply the preceding lemma to the right multiplication map φ_a: $x \longmapsto xa$.

§ 4

Definitions. When S is a semigroup and $ab = a$ holds for any $a, b \epsilon S$, S is called a left zero semigroup. The following fact is well-known (for instance, [7] pp. 77-80). A semigroup which satisfies such equivalent conditions is called a left group.

Lemma 4.1 When S is a semigroup, the following three conditions are equivalent.

(1)(i) S has a right identity, and (ii) for any $a \epsilon S$ and any right identity $e \epsilon S$, there exists $x \epsilon S$ such that $xa = e$.

(2) For any $a, b \epsilon S$, there exists a unique $x \epsilon S$ such that $xa = b$.

(3) S is isomorphic to the direct product of a group and a left zero semigroup.

Now we can state the following:

Theorem 4.2 (1) If $S(R) = T(R) \neq \phi$, then S(R) is a left group. Hence, if R is both left Noetherian and left Artinian, S(R) coincides with T(R) and is a left group unless it is empty.

(2) When R is both left Noetherian and left Artinian, R has a right identity if and only if $S(R) \neq \phi$.

Proof. (1) We shall show that S(R) satisfies (2) of Lemma 4.1. Let $a, b \epsilon S(R)$. Since $Ra = R \epsilon b$, there exists $x \epsilon R$ such that $xa = b$. We have to show that $x \epsilon S(R)$. If $x \epsilon S(R)$, there exists a non-zero element $y \epsilon R$ such that $yx = 0$, for $S(R) = T(R)$. Then $yxa = yb = 0$, hence $A_l(b) \neq 0$, which contradicts $b \epsilon S(R) = T(R)$. So $x \epsilon S(R)$. Next assume that $xa = b$ and $x'a = b$. Then $(x - x')a = 0$, which follows $x = x'$, since $x - x' \epsilon A_l(a) = 0$. Thus S(R) is a left group.

(2) Suppose $S(R) = T(R) \neq \phi$, then it is a left group, hence has a right identity e by Lemma 4.1. Since $Re = R$, e is a right identity of R.

Corollary 4.3 If R has no left ideals other than 0 and R, then R is either a division ring or a zero ring on a cyclic group of prime order.

Proof. If $R^2 = 0$, then the additive group of R is a cyclic group of prime order since it is a simple Abelian group. So we can suppose there exists $a \in R$ such that $Ra = R$. By Theorem 4.2 R has a right identity, so R has an identity by Corollary 1.3. It is immediate that R is a division ring.

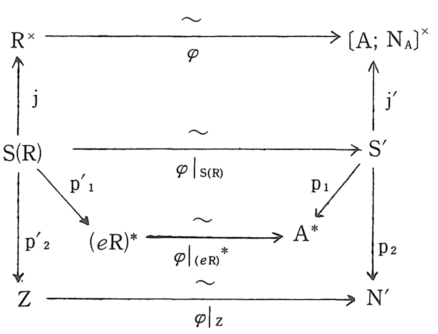
Let R be a ring such that $S(R) = T(R) \neq \phi$, then S(R) must be isomorphic to the direct product of a group and a left zero semigroup. Let e be a right identity of R and put $A = eR$ and $N = A_r(R)$, then there exists an isomorphism $\varphi: R \longrightarrow [A; N_A]$. If we identify R with $[A; N_A]$ by φ, then we can write any element of R as (a,n), where $a \in A$ and $n \in N$. Suppose $R \ni s = (a,n)$ satisfies $Rs = R$, then there exist $b \in A$ and $n' \in N$ such that $(b,n')(a,n) = (ba, n'a) = (e,0)$, which follows that $ba = e$. Conversely, let n be an arbitrary element of N and $a \in A$ satisfy $ba = e$ *for some* $b \in A$. Then for any element(c,m) of R it holds that $(cb,mb)(a,n) = (c,m)$, so $s = (a,n)$ satisfies $Rs = R$. Hence, if we put $A' = \{a \in A \mid ba = e \text{ for some } b \in A\}$, $(a,n) \in S(R)$ is equivalent to $a \in A'$.

Let a be an arbitrary element of A'. As $(a,0) \in S(R)$, by Lemma 4.1 (2), there exist $a' \in A'$ and $n \in N$ such that $(a',n)(a,0) = (a'a, na) = (e,0)$. It follows that $a'a = e$. On the other hand,

$$\begin{aligned}(a,0)(a',0)(a,0) &= (aa',0)(a,0)\\ &= (e,0)(a,0).\end{aligned}$$

Hence $aa' = e$ by the uniqueness of Lemma 4.1 (2). So A' is nothing but the unit group A^* of A.

Let us put $N' = \{(1,n) \mid n \in N\}$ and define $p_2: S' = \{(a,n) \mid a \in A^*, n \in N\} \longrightarrow N'$ by $(a,n) \longmapsto (1,n)$. $p_1: S' \longrightarrow A^*$ is defined by $(a,n) \longmapsto a$. Thus we get the following commutative diagram of semigroups:



Here $(eR)^*$ denotes the unit group of eR, and Z the left zero semigroup consisting of all right identities of R. j and j' are natural injections. $p'_1 = (\varphi \mid_{(eR)^*})^{-1} \circ p_1 \circ (\varphi \mid_{S(R)})$, $p'_2 = (\varphi \mid_Z)^{-1} \circ p_2 \circ (\varphi \mid_{S(R)})$. p_1 and p_2 are orthogonal (cf. [7] pp. 76-77). For, let $\Delta_1: S' = \bigcup_{b \in A^*} U_b$ be the partition of S' induced by p_1, where $U_b = \{(b,n) \mid n \in N\}$. Also let $\Delta_2: S' = \bigcup_{m \in N} V_m$ be the partition induced by p_2, where $V_m = \{(a,m) \mid a \in A^*\}$. Then clearly $U_b \cap V_m$ consists of only one element (b,m). So Δ_1 and Δ_2 are orthogonal. Consequently S' is isomorphic to the direct product of A^* and N'.

p'_1 and p'_2 are orthogonal, too, so S(R) is isomorphic to the direct product of $(eR)^*$ and Z. Note that A^* is isomorphic to the unit group of the left R-endomorphism ring of R by Lemma 1.1 (3). So we get the following:

Theorem 4.4 If $S(R) = T(R) \neq \phi$, then S(R) is isomorphic to the direct product of the unit group of the left R-endomorphism ring of R and the left zero semigroup consisting of all right identities of R.

Note that if R is left Artinian moreover, then Z is a finite set by Theorem 1.5.

Theorem 4.5 If R is both left Noetherian and left Artinian, then the following three conditions are equivalent.

(1) R has a right identity.

(2) There exists $a \in R$ such that $Ra = R$.

(3) For any $a \in R$, there exists $b \in R$ such that $ab = a$.

Proof. Clear from Theorem 2.4 and Theorem 4.2 (2).

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