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A Remark on Random Walks on Z^d

Z^d 上のランダムウォークに関する一注意

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Abstract. We consider the random walks on the state space Z^d. It is well known that the simple random walk is recurrent for $d = 1, 2$ and transient for $d \geqq 3$ and that the non-symmetric Bernoulli random walk is transient for any d. In this paper, we consider the random walks of mixed type which are not spatially homogeneous and give the examples of recurrent random walks of this type.

§ 1. Let Z^d be the space of d-dimensional integers, that is, the set of lattice points

$$x = (i_1, \cdots, i_d),\ \ (i_k \text{ is an integer for } k = 1, \cdots, d).$$

We consider a particle starting at the origin of Z^d and moving from x to y in Z^d at each step with the probability $p(x, y)$ satisfying the following conditions

$$p(x, y) \geqq 0, \qquad \sum_{y \in Z^d} p(x, y) = 1.$$

The motion of this particle is called a random walk on the state space Z^d with the transition probabilities $\{p(x, y)\}$.

Let $S_0, S_1, S_2, \cdots$ be the sequence of random variables whose ranges are contained in Z^d. The sequence is called a Markov chain if

$$P[S_{n+1} = y \,|\, S_0 = x_0, S_1 = x_1, \cdots, S_{n-1} = x_{n-1}, S_n = x] = P[S_{n+1} = y \,|\, S_n = x]$$

for every n and $x_0, x_1, \cdots, x_{n-1}$ in Z^d, where $P[A]$ denotes the probability occurring of the event A. The random walk starting at the origin 0 is the Markov chain with $S_0 = 0$ and $P[S_{n+1} = y \,|\, S_n = x] = p(x, y)$.

We consider the random walk on Z^d and set

$$f_{2n} = P[S_0 = 0, S_2 \neq 0, \cdots, S_{2n-2} \neq 0, S_{2n} = 0].$$

The probability f_{2n} is the probability of the return to the origin for the first time after $2n$ steps. Therefore,

$$f = \sum_{n=1}^{\infty} f_{2n}$$

is the probability of the return to the origin infinitely often. When $f = 1$, the random walk is called recurrent and when $f < 1$, the the random walk is called transient.

The random walk on Z with the transition probabilities

$$p(i, i+1) = p, \quad p(i, i-1) = q, \quad p(i, j) = 0\ (j \neq i+1, i-1) \quad (0 < p < 1, p + q = 1)$$

is called the Bernoulli random walk. When $p = q$, the walk is called symmetric and when $p \neq q$, the walk is called non-symmetric. The symmetric Bernoulli random walk is usually called the simple random walk. The symmetric random walk is recurrent and the non-symmetric random walk is transient.

In this paper, we shall consider the recurrence or transience property of the mixed random walks of these two types. The method used here is the combinatorial one by counting the number of paths of the random walk, which is found in Feller[1].

§ 2. We consider a random walk on Z. Let $S_0, S_1, S_2, \cdots$ be the sequence of random variables of this random walk. We plot the points $(0, 0), (1, S_1), \cdots, (n, S_n)$ in tx-plane and connect these points by the line segments successively. We call it the path of length n.

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We denote by $L[A]$ be the number of paths satisfying the condition A. The number of paths from $(0, 0)$ to $(0, 2n)$ is denoted by U_{2n} and the number of paths from $(0, 0)$ to $(0, 2n)$ without touching or crossing the t-axis is denoted by F_{2n}. It is evident that $U_{2n} = {}_{2n}C_n$. For F_{2n}, we have the following proposition.

Proposition 1

$F_{2n} = \frac{1}{2n-1}\ {}_{2n}C_n$

Proof

The proof is found in Feller[1]. For the sake of completeness, we give the proof.

$$\begin{aligned} F_{2n} &= 2\,L[S_1 = 1, S_2 > 0, \cdots, S_{2n-2} > 0, S_{2n-1} = 1] \\ &= 2\,\{\,L[S_1 = 1, S_{2n-1} = 1] - L[S_1 = 1, S_k = 0, S_{2n-1} = 1]\,\} \quad \text{(for some } k) \end{aligned}$$

Here we have $L[S_1 = 1, S_k = 0, S_{2n-1} = 1] = L[S_1 = -1, S_{2n-1} = 1]$ by the reflection principle.

$$\begin{aligned} F_{2n} &= 2\,\{\,L[S_1 = 1, S_{2n-1} = 1] - L[S_1 = -1, S_{2n-1} = 1]\,\} \\ &= 2\,\{\,{}_{2n-2}C_{n-1} - {}_{2n-2}C_n\,\} = 2\,(1 - \tfrac{n-1}{n})\ {}_{2n-2}C_{n-1} \\ &= \tfrac{2}{n}\ {}_{2n-2}C_{n-1} = \tfrac{1}{2n-1}\ {}_{2n}C_n \end{aligned}$$

Q.E.D.

We consider the random walk with transition probabilities

$$p(0, 1) = p, \quad p(0, -1) = q \quad (p + q = 1, p > q),$$
$$p(i, i + 1) = p(i, i - 1) = \tfrac{1}{2} \quad (i \neq 0)$$

and call it the random walk of type I. Further we consider the random walk with transition probabilities

$$p(0, 1) = p(0, -1) = \tfrac{1}{2},$$
$$p(i, i + 1) = q, \quad p(i, i - 1) = p \quad (i > 0), \quad p(i, i + 1) = p, \quad p(i, i - 1) = q \quad (i < 0), \quad (p + q = 1, p > q)$$

and call it the random walk of type II.

Theorem 1

The random walks on Z of type I and of type II are recurrent.

Proof

For the type I, the probability of the return to the origin for the first time after $2n$ steps is given by Proposition 1.

$$\begin{aligned} f_{2n} &= P[S_0 = 0, S_2 \neq 0, \cdots, S_{2n-2} \neq 0, S_{2n} = 0]. \\ &= \tfrac{1}{2}\,\{P[S_0 = 0, S_1 > 0, \cdots, S_{2n-1} > 0, S_{2n} = 0] + P[S_0 = 0, S_1 < 0, \cdots, S_{2n-1} < 0, S_{2n} = 0]\,\} \\ &= \tfrac{1}{2}\,\{\tfrac{1}{2n-1}\ {}_{2n}C_n(\tfrac{1}{2}p)(\tfrac{1}{2})^{2n-2} + \tfrac{1}{2n-1}\ {}_{2n}C_n(\tfrac{1}{2}q)(\tfrac{1}{2})^{2n-2}\,\} \\ &= \tfrac{1}{2n-1}\ {}_{2n}C_n(\tfrac{1}{2})^{2n} \end{aligned}$$

According to the recurrence of the simple random walk,

$$f = \sum_{n=1}^{\infty} f_{2n} = \sum_{n=1}^{\infty} \tfrac{1}{2n-1}\ {}_{2n}C_n(\tfrac{1}{2})^{2n} = 1.$$

For the type II, the probability of the return to the origin for the first time after $2n$ steps is given by Proposition 1.

$$\begin{aligned} f_{2n} &= P[S_0 = 0, S_2 \neq 0, \cdots, S_{2n-2} \neq 0, S_{2n} = 0]. \\ &= \tfrac{1}{2}\,\{P[S_0 = 0, S_1 > 0, \cdots, S_{2n-1} > 0, S_{2n} = 0] + P[S_0 = 0, S_1 < 0, \cdots, S_{2n-1} < 0, S_{2n} = 0]\,\} \\ &= \tfrac{1}{2}\,\{\tfrac{1}{2n-1}\ {}_{2n}C_n(\tfrac{1}{2}p)(pq)^{n-1} + \tfrac{1}{2n-1}\ {}_{2n}C_n(\tfrac{1}{2}p)(pq)^{n-1}\,\} \\ &= \tfrac{1}{2q}\tfrac{1}{2n-1}\ {}_{2n}C_n(pq)^n \end{aligned}$$

According to the transience of the non-symmetric Bernoulli random walk,

$$f = \sum_{n=1}^{\infty} f_{2n} = \tfrac{1}{2q}\sum_{n=1}^{\infty} \tfrac{1}{2n-1}\ {}_{2n}C_n(pq)^n = \tfrac{1}{2q}(1 - \sqrt{1 - 4pq}\,) = \tfrac{1}{2q}(2q) = 1.$$

Q.E.D

§3. We consider the random walk on Z^2. Let $S_0, S_1, S_2, \cdots$ be the sequence of vector valued random variables of this random walk. We plot the points $(0,(0,0)),(1,S_1),\cdots,(n,S_n)$ in the txy-space and connect these points by the line segments successively. We call it the path of length n.

The number of paths from $(0,(0,0))$ to $(2n,(0,0))$ is denoted by U_{2n} and the number of paths from $(0,(0,0))$ to $(2n,(0,0))$ without touching or crossing the t-axis is denoted by F_{2n}. It is shown that $U_{2n} = ({}_{2n}C_n)^2$. But, the proposition similar to Proposition 1 cannot be shown explicitly. Nevertheless, the theorem similar to Theorem 1 can be shown as well.

As in § 2, we define the random walks of type I and type II. For the type I, the transition probabilities are given as follows.

$$p((i,0),(i,1)) = \tfrac{p}{2}, \quad p((i,0),(i,-1)) = \tfrac{q}{2}, \quad (p+q=1, p>q)$$
$$p((i,j),(i,j+1)) = p((i,j),(i,j-1)) = \tfrac{1}{4} \quad (j \neq 0)$$
$$p((i,j),(i+1,j)) = p((i,j),(i-1,j)) = \tfrac{1}{4}$$

For the type II, the transition probabilities are given as follows.

$$p((i,0),(i,1)) = p((i,0),(i,-1)) = \tfrac{1}{4}$$
$$p((i,j),(i,j+1)) = \tfrac{q}{2}, \quad p((i,j),(i,j-1)) = \tfrac{p}{2} \quad (j>0), \quad (p+q=1, p>q)$$
$$p((i,j),(i,j+1)) = \tfrac{p}{2}, \quad p((i,j),(i,j-1)) = \tfrac{q}{2} \quad (j<0), \quad (p+q=1, p>q)$$
$$p((i,j),(i+1,j)) = p((i,j),(i-1,j)) = \tfrac{1}{4}$$

Theorem 2

The random walk on Z^2 of type I is recurrent.

Proof

Let the number of paths returning to the origin for the first time after $2n$ steps be F_{2n}. We take a path of this type. Let the number of paths above or below the tx-plane be k, with k_1 above and k_2 below $(k_1+k_2=k)$. The probability of this path is

$$(\tfrac{1}{4}\tfrac{p}{2})^{k_1}(\tfrac{1}{4}\tfrac{q}{2})^{k_2}(\tfrac{1}{4})^{2n-2k}.$$

Considering the reflection on tx-plane, there are 2^k paths of this type. The probability of these paths is

$$\sum_{k_1+k_2=k} {}_kC_{k_1}(\tfrac{1}{4}\tfrac{p}{2})^{k_1}(\tfrac{1}{4}\tfrac{q}{2})^{k_2}(\tfrac{1}{4})^{2n-2k}$$
$$= (\tfrac{1}{4})^{k_1+k_2}(\tfrac{p}{2}+\tfrac{q}{2})^k(\tfrac{1}{4})^{2n-2k} = (\tfrac{1}{4})^{2n-k}(\tfrac{1}{2})^k = (\tfrac{1}{4})^{2n}2^k.$$

Therefore we have

$$f_{2n} = F_{2n}(\tfrac{1}{4})^{2n}.$$

According to the recurrence of the simple random walk on Z^2,

$$f = \sum_{n=1}^{\infty} f_{2n} = \sum_{n=1}^{\infty} F_{2n}(\tfrac{1}{4})^{2n} = 1. \qquad \text{Q.E.D}$$

For the random walk on Z^2 of type II, the calculation of f is not completed yet. It is probable that this random walk will be recurrent. As for the random walks on Z^3, we can prove in the same way that the random walk of type I is transient. However, the recurrence or transience of the random walk of type II is not known.

§4. We further investigate the random walk on Z of type I. We consider the number of paths from $(0,0)$ to $(0,2n)$ touching or crossing the t-axis for k times without including $(0,0)$ and denote it by $F_{2n}^{(k)}$.

Proposition 2

$$F_{2n}^{(k)} = \tfrac{k}{2n-k}\ {}_{2n-k}C_n 2^k$$

Proof

We denote $L_{2n-k,k}$ the number of paths from $(0,0)$ to $(2n-k,k)$.

$$F_{2n}^{(k)} = L_{2n-k,k}\, 2^k = \tfrac{k}{2n-k}\ {}_{2n-k}C_n\, 2^k \qquad \text{Q.E.D}$$

Next we consider the number of paths of length $2n$ touching or crossing the t-axis for k times without including $(0,0)$ and denote it by $H_{2n}^{(k)}$.

Proposition 3

$H_{2n}^{(k)} = {}_{2n-k}C_n\, 2^k$

Proof

We can represent $H_{2n}^{(k)}$ by $F_{2n}^{(k)}$.

$$\begin{aligned} H_{2n}^{(k)} &= F_{2n}^{(k)} + F_{2n}^{(k+1)} + \cdots + F_{2n}^{(n)} \\ &= ({}_{2n-k}C_n 2^k - {}_{2n-k-1}C_n 2^{k+1}) + ({}_{2n-k-1}C_n 2^{k+1} - {}_{2n-k-2}C_n 2^{k+2}) + \cdots + {}_{2n-n}C_n 2^n \\ &= {}_{2n-k}C_n 2^k \end{aligned}$$

Q.E.D

We consider the probability of the return to the origin up to and including $2n$ steps for k times and denote it by $h_{2n}^{(k)}$.

Theorem 3

$h_{2n}^{(k)} = {}_{2n-k}C_n\, (\frac{1}{2})^{2n-k}$

Proof

We take a path of type $F_{2n}^{(k)}$, then the number of paths above or below the t-axis is k, with k_1 above and k_2 below ($k_1 + k_2 = k$). The probability of this path is

$$(\tfrac{1}{2}p)^{k_1}(\tfrac{1}{2}q)^{k_2}(\tfrac{1}{2})^{2n-2k}$$

Adding these terms for 2^k paths of reflection on the t-axis, we have

$$\sum_{k_1+k_2=k} {}_kC_{k_1}(\tfrac{1}{2}p)^{k_1}(\tfrac{1}{2}q)^{k_2}(\tfrac{1}{2})^{2n-2k}$$
$$= (\tfrac{1}{2})^{k_1+k_2}(p+q)^k(\tfrac{1}{2})^{2n-2k} = (\tfrac{1}{2})^{2n-k} = (\tfrac{1}{2})^{2n}2^k.$$

For the paths of the other types, the number of paths above or below the t-axis is $k+1$, we have a similar equality. Therefore,

$$h_{2n}^{(k)} = {}_{2n-k}C_n\, 2^k\, (\tfrac{1}{2})^{2n} = {}_{2n-k}C_n\, (\tfrac{1}{2})^{2n-k}.$$

Q.E.D

Theorem 1 is the same as in the case of the simple random walk. Therefore, we have

$$h_{2n}^{(1)} > h_{2n}^{(2)} > \cdots > h_{2n}^{(n)},$$

which shows that the probability of the return to the origin for k times after $2n$ steps decreases as k increases.

References

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